QUASITRIANGULAR COIDEAL SUBALGEBRAS OF $U_q(g)$ IN TERMS OF GENERALIZED SATAKE DIAGRAMS

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Abstract. Let $g$ be a finite-dimensional semisimple complex Lie algebra and $\theta$ an involutive automorphism of $g$. According to Letzter, Kolb and Balagović the fixed-point subalgebra $t = g^\theta$ has a quantum counterpart $B$, a coideal subalgebra of the Drinfeld-Jimbo quantum group $U_q(g)$ possessing a universal K-matrix $K$. The objects $\theta$, $t$, $B$ and $K$ can all be described in terms of combinatorial datum, a Satake diagram. In the present work we extend this construction to generalized Satake diagrams, objects first considered by Heck. A generalized Satake diagram naturally defines a semisimple automorphism $\theta$ of $g$ restricting to the standard Cartan subalgebra $h$ as an involution. It also defines a subalgebra $k$ containing $t$ satisfying $t \cap h = h^\theta$, but not necessarily a fixed-point subalgebra. The subalgebra $t$ can be quantized to a coideal subalgebra of $U_q(g)$ endowed with a universal K-matrix in the sense of Kolb and Balagović. We conjecture that all such coideal subalgebras of $U_q(g)$ arise from generalized Satake diagrams in this way.

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1. Introduction

Given a finite-dimensional semisimple complex Lie algebra $g$ and an involutive Lie algebra automorphism $\theta \in \text{Aut}(g)$, a symmetric pair is a pair $(g, t)$ where $t = g^\theta$ is the $\theta$-fixed subalgebra of $g$, see [Ar62, Sa71]. Quantum symmetric pairs are their quantum analogons. That is to say, the enveloping algebra $U(g)$ can be quantized to a quasitriangular Hopf algebra, the Drinfeld-Jimbo quantum group $U_q(g)$ endowed with the universal R-matrix $R$, see [Ji85, Dr87]. Similarly, the $\theta$-fixed subalgebra $t$ can be quantized to a coideal subalgebra $B \subseteq U_q(g)$ [Le99, Le02, Ko14] having a compatible quasitriangular structure, the universal K-matrix $K$ [BK16, Ko17] (see also [BW13, Sec. 2.5] for the case of quantum symmetric pairs of type AIII/AIV). Quantizations of symmetric pairs appeared earlier in a rather different approach in [NDS95, NS95] (also see [KS09]). An earlier notion of a universal K-matrix, not directly linked to a quantum symmetric pair, appeared in [DKM03].

The map $\theta$, the fixed-point subalgebra $t$, the coideal subalgebra $B$ and the universal object $K$ are all defined in terms of combinatorial information, the so-called Satake diagram $(X, \tau)$. Here $X$ is a subdiagram of the Dynkin diagram of $g$ and $\tau$ is an involutive diagram automorphism stabilizing $X$ and satisfying certain compatibility conditions, see [Le02, Ko14].

It is the aim of this paper to extend some of the above work to a more general setting than (quantizations of) fixed-point subalgebras. A direct motivation for this is the fact that the correct quantum group analogue of the fixed-point subalgebra in the Letzter-Kolb theory is not a fixed-point subalgebra itself, but merely...
tends to one as \( q \to 1 \), see [Le99, Sec. 4] and [Ko14, Ch. 10]. This suggests that there may be a generalization of this theory that does not require a fixed-point subalgebra as input.

A careful analysis of [Ko14, BK15, BK16] indeed indicates that the compatibility conditions for \( X \) and \( \tau \) can be weakened. Indeed, in [BK15, Rmks. 2.6, 3.14] it is explicitly suggested that some key passages of the theory are amenable for generalizations. This leads to the notion of a generalized Satake diagram, see Definition 2.2, and the whole theory survives in this setting with minor adjustments. The resulting Lie subalgebra \( \mathfrak{t} = \mathfrak{t}(X, \tau) \) is given in Definition 3.1 and the corresponding coideal subalgebra \( B = B(X, \tau) \) in Definition 4.1. For \( \mathfrak{g} \) of type \( \Lambda \), all generalized Satake diagrams are Satake diagrams. For other \( \mathfrak{g} \), the generalized Satake diagrams that are not Satake diagrams are listed in Table 1.

Our proposed generalization of Satake diagrams can be traced back to the work of A. Heck [He84]. In this work Heck provides a classification of involutions of finite root systems such that the corresponding restricted Weyl group is the Weyl group of the restricted root system. We will review this approach and connect with a theorem of Lusztig stating that the restricted Weyl group is in fact a Coxeter group. The characterization in terms of the restricted Weyl group is relevant in the context of the universal \( R \) - and \( K \)-matrices for quantum symmetric pairs. The universal \( R \)-matrix \( R \) has a distinguished factor called quasi \( R \)-matrix playing an important role in the theory of canonical bases for \( U_q(\mathfrak{g}) \) developed by Kashiwara and Lusztig, see [Ka90] and [Lu94, Part IV]. The quasi \( R \)-matrix possesses a remarkable factorization property expressed in terms of the braid group action on \( U_q(\mathfrak{g}) \) of the Weyl group associated to \( \mathfrak{g} \), see e.g. [KR90, LS90]. Recently it has become clear that many of these properties extend to the universal \( K \)-matrix \( K \). It has a distinguished factor called quasi \( K \)-matrix introduced in [BW13] for certain coideal subalgebras of \( U_q(\mathfrak{sl}_N) \) and in a more general setting in [BK15], and featuring prominently in the theory of canonical bases for quantum symmetric pairs [BW16]: for a historical note we refer the reader to [BW16, Rmk. 4.9]. In [DK18] a factorization property is established for the quasi \( K \)-matrix using a braid group action of the restricted Weyl group. As a consequence of the present work, this factorization property naturally extends to quasi \( K \)-matrices defined in terms of generalized Satake diagrams.

A generalization of this approach to the Kac-Moody setting will be addressed in a future work. Another outstanding issue is a Lie-theoretic motivation of the subalgebra \( \mathfrak{t} \) which we define in a rather ad hoc manner directly in terms of the combinatorial data \( (X, \tau) \), see Definition 3.1. Therefore we now provide an additional motivation for the study of the subalgebra \( \mathfrak{t} \) and its quantization \( B \).

### 1.1. Some remarks on the representation theory of \((U_q(\mathfrak{g}), B)\). \label{1.1}

There exists a completion \( \mathcal{U} \) of \( U_q(\mathfrak{g}) \) and a completion \( \mathcal{U}^{(2)} \) of \( U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \) in which \( \mathcal{U} \otimes \mathcal{U} \) can be embedded; these are completions with respect to the category of integrable \( U_q(\mathfrak{g}) \)-modules, so that objects in them have well-defined images under any finite-dimensional representation, see e.g. [Lu94, Jan96]. In particular, one can construct an invertible \( \mathcal{R} \in \mathcal{U}^{(2)} \) satisfying

\[
\mathcal{R}\Delta(a) = \Delta^{op}(a)\mathcal{R} \quad \text{for all } a \in U_q(\mathfrak{g}), \quad (\Delta \otimes \text{id})(\mathcal{R}) = R_{13}R_{23}, \quad (\text{id} \otimes \Delta)(\mathcal{R}) = R_{13}R_{12},
\]

where \( \Delta \) is the coproduct and \( \Delta^{op} \) the opposite coproduct; these can be viewed as maps from \( \mathcal{U} \) to \( \mathcal{U}^{(2)} \).

Analogously, according to [BK16, Ko17], one can construct an invertible \( \mathcal{K} \in \mathcal{U} \) and an involutive Hopf algebra automorphism \( \phi \) of \( \mathcal{U} \) such that \( (\phi \otimes \phi)(\mathcal{R}) = \mathcal{R} \) and

\[
\begin{align*}
\mathcal{K}b &= \phi(b)\mathcal{K} \quad \text{for all } b \in B, \\
(\mathcal{R}^\phi)_{21}\mathcal{K}_2\mathcal{R} &\in \mathcal{B}^{(2)}, \\
\Delta(\mathcal{K}) &= \mathcal{R}_{21}(1 \otimes \mathcal{K})\mathcal{R}^\phi(\mathcal{K} \otimes 1),
\end{align*}
\]

where \( \mathcal{R}^\phi = (\phi \otimes \text{id})(\mathcal{R}) \), the subscript \( 21 \) denotes the simple transposition of tensor factors in \( \mathcal{U}^{(2)} \) and \( \mathcal{B}^{(2)} \subseteq \mathcal{U}^{(2)} \) is a particular completion of \( B \otimes U_q(\mathfrak{g}) \), see [Ko17, Eq. (3.31)]. As a consequence of the above properties, the (universal) \( \phi \)-twisted reflection equation is satisfied:

\[
\mathcal{R}_{21}(1 \otimes \mathcal{K})\mathcal{R}^\phi(\mathcal{K} \otimes 1) = (\mathcal{K} \otimes 1)(\mathcal{R}^\phi)_{21}(1 \otimes \mathcal{K})\mathcal{R} \in \mathcal{U}^{(2)}.
\]
The automorphism $\phi$ is given by $\tau \tau_0$ where $\tau_0$ is the diagram automorphism corresponding to the longest element of the Weyl group of $\mathfrak{g}$. The expression for $K$ is given in [BK16, Cor. 7.7].

One could argue in favour of making the automorphism $\phi$ inner: adjoin to $\mathcal{U}$ a group-like element $c_{\phi}$ such that $\phi(u) = c_{\phi}uc_{\phi}^{-1}$ for all $u \in \mathcal{U}$. Then the object $\tilde{K} := c_{\phi}Kc_{\phi}^{-1}$ satisfies (1.1-1.3) with $\phi$ replaced by id. However, for certain nontrivial diagram automorphisms $\phi$, $c_{\phi}$ cannot be chosen inside $\mathcal{U}$ so that $\tilde{K}$ cannot be evaluated in all finite-dimensional representations. For instance, if $\phi$ is the vector representation of $U_q(\mathfrak{sl}_N)$ with $N > 2$ one checks that the matrices $\rho(\phi(u))$ and $\rho(u)$ are not simultaneously similar for all $u \in U_q(\mathfrak{g})$. This relates to the fact that the weights defining certain fundamental representations are not fixed by $\phi$.

Now let $\mathfrak{g}$ be any finite-dimensional semisimple complex Lie algebra and $\rho$ the vector representation of $U_q(\mathfrak{g})$; if $\mathfrak{g}$ is of exceptional type by this we mean the smallest fundamental representation (for $E_6$ one has a choice of two representations). Choose $R \in \text{GL}(V \otimes V)$ proportional to $(\rho \otimes \rho)(R)$, $R^\phi \in \text{GL}(V \otimes V)$ proportional to $(\rho \otimes \rho)(R^\phi)$ and $K \in \text{GL}(V)$ proportional to $\rho(K)$. Applying $\rho \otimes \rho$ to (1.4) one obtains the matrix reflection equation

\[(1.5) \quad R_{21}(\text{Id} \otimes K) R^\phi(K \otimes \text{Id}) = (K \otimes \text{Id})(R^\phi)_{21}(\text{Id} \otimes K)R \quad \text{in } \text{End}(V \otimes V)\]

where the subscript $21$ indicates conjugation by the permutation operator $P \in \text{GL}(V \otimes V)$. Starting with $\mathfrak{g}$ of classical Lie type and a coideal subalgebra $B = B(X, \tau)$ where $(X, \tau)$ is a Satake diagram, the matrices $\rho(K)$ recover the solutions of (1.5) used in [NDS95, NS95] to define quantum symmetric pairs.

Treating the matrix $R$ as given, one can of course solve (1.5) for $K \in \text{GL}(V)$. For $U_q(\mathfrak{sl}_N)$ and $V = \mathbb{C}^N$ this was done by A. Mudrov [Mu02]. From this result and computations for $U_q(\mathfrak{g})$ whose vector representation is of dimension at most 9 (i.e. with $\mathfrak{g}$ of types $B_n$, $C_n$, $D_n$, $n \leq 4$ and $G_2$) one obtains a classification of solutions $K$ of (1.5) for those pairs $(U_q(\mathfrak{g}), \rho)$. One can match this list of solutions $K$ one-to-one with a list of generalized Satake diagrams $(X, \tau)$ by checking which $K$ satisfies $K \rho(b) = \rho(\phi(b))K$ for all $b \in B = B(X, \tau)$, i.e. the image of (1.1) under $\rho$. Although this intertwining equation does not determine $K$ uniquely, it turns out that, provided $K \notin \mathbb{C}I_d$, each $K$ intertwines $\rho|_B$ for a unique $B = B(X, \tau)$ with $X$ not equal to the underlying Dynkin diagram $I$. In the case $X = I$ we must have $\tau = \tau_0$ and $B = U_q(\mathfrak{g})$; naturally it can be matched to the excluded case $K \in \mathbb{C}I_d$. It leads to the following conjecture.

**Conjecture 1.1.** Let $\rho : U_q(\mathfrak{g}) \to \text{End}(V)$ be the vector representation of $U_q(\mathfrak{g})$.

(i) If $K \in \text{GL}(V)$ is a solution of (1.5) there exists a generalized Satake diagram $(X, \tau)$ such that $K$ is proportional to $\rho(K)$ where $K = K(X, \tau)$ is the universal $K$-matrix for the subalgebra $B = B(X, \tau)$.

(ii) The representation $\rho$ can be used to distinguish quasitriangular coideal subalgebras: if $(B, K)$, $(B', K')$ are distinct quasitriangular coideal subalgebras of $U_q(\mathfrak{g})$ then $\rho(K)$ and $\rho(K')$ are not proportional.

Hence the only quasitriangular coideal subalgebras of $U_q(\mathfrak{g})$ are of the form $(B(X, \tau), K(X, \tau))$ with $(X, \tau)$ a generalized Satake diagram.

In the Letzter-Kolb approach, the generators of the coideal subalgebra $B$ associated to a node $i \in I \setminus X$ carry extra parameters: the scalars $\gamma_i \neq 0$ and $\sigma_i$, see Definition 4.1. We can sharpen part (i) of Conjecture 1.1: any invertible matrix solution $K$ of (1.5) is proportional to $\rho(K)$ for some $B(X, \tau)$ with $(X, \tau)$ a generalized Satake diagram and the parameters satisfying certain constraints. Typical constraints were found in [Le03, Ko14] and are given in terms of the sets $\Gamma_q$ and $\Sigma_q$, see (4.3). More generally, we must have $(\gamma_i)_{i \in I \setminus X} \in \Gamma_q$. For the constraints on $\sigma_i$ we consider $I_{ns} = \{ i \in I \setminus X \mid \text{do not neighbour } X, \tau(i) = \emptyset \}$, see (3.17). If $i \notin I_{ns}$ then $\sigma_i = 0$. For all $(i, j) \in I_{ns} \times I_{ns}$ such that $i \neq j$ conjecturally one of three conditions must hold: the Cartan integer $a_{ij}$ is even, $\sigma_j = 0$, or $\sigma_j^2/\gamma_i$ lies in a particular finite subset of a quadratic completion of $\mathbb{C}(q)$. The set $\Sigma_q$ does not cover the third possibility, which appeared in [BB10].

**1.2. Outline.** The paper is organized as follows. In Section 2 we define the basic objects associated to a finite-dimensional semisimple complex Lie algebra $\mathfrak{g}$ and its Cartan subalgebra $\mathfrak{h}$. We introduce generalized Satake diagrams and explain how they emerge in the work of A. Heck.

In Section 3 we define the subalgebra $\mathfrak{t} = \mathfrak{t}(X, \tau) \subseteq \mathfrak{g}$. Theorem 3.2 is the main result of this section. We show that $\mathfrak{t}$ satisfies the intersection condition $\mathfrak{t} \cap \mathfrak{h} = \mathfrak{h}^0$ (which trivially holds when $\mathfrak{t} = \mathfrak{g}^0$ with $\theta^2 = \text{id}_\mathfrak{g}$).
precisely if \((X, \tau)\) is a generalized Satake diagram. We then study the derived subalgebra of \(\mathfrak{k}\). When \(\mathfrak{k}\) is not a reductive Lie algebra, Propositions 3.4 and 3.7 establish a semidirect product decomposition for \(\mathfrak{k}\) in terms of a reductive subalgebra and a nilpotent ideal of class 2. We end this section with some results about the universal enveloping algebra \(U(\mathfrak{k})\). (Appendix A contains three technical lemmas in aid of Section 3.)

In Section 4 we indicate the necessary modifications to the papers [Ko14, BK15, BK16, Ko17, DK18] so that they apply to the quantum pair algebras \(B = U_q(\mathfrak{k})\) associated to generalized Satake diagrams.

We use the symbol \(\g\) to indicate the end of definitions, examples and remarks.

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2. Finite-dimensional semisimple Lie algebras and root system involutions

Let \(I\) be a finite set and \(A = (a_{ij})_{i,j \in I}\) a Cartan matrix. In particular, there exist positive rationals \(d_i\) \((i \in I)\) such that \(d_i a_{ij} = d_j a_{ji}\). Let \(g = g(A)\) be the corresponding finite-dimensional semisimple Lie algebra over \(\mathbb{C}\). It is generated by \(\{e_i, f_i, h_i\}_{i \in I}\) subject to

\[
\begin{align*}
[&h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j, \quad [e_i, f_j] = \delta_{ij}h_i \\
&\text{ad}(e_i)^{1-a_{ij}}(e_j) = \text{ad}(f_i)^{1-a_{ij}}(f_j) = 0 \quad \text{if } i \neq j,
\end{align*}
\]

for all \(i, j \in I\). We denote the standard Cartan subalgebra by \(\mathfrak{h} = \langle h_i \mid i \in I \rangle\) and also consider the corresponding nilpotent subalgebras \(n^+ = \langle e_i \mid i \in I \rangle\), \(n^- = \langle f_i \mid i \in I \rangle\).

The simple roots \(\alpha_i \in \mathfrak{h}^*\) \((i \in I)\) satisfy \(\alpha_j(h_i) = a_{ij}\) for \(i, j \in I\). Let \(Q = \sum_{i \in I} \mathbb{Z}a_i\) denote the root lattice and write \(Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}a_i\). In terms of the root spaces \(g_\alpha = \{x \in g : \forall h \in \mathfrak{h}, [h, x] = \alpha(h)x\}\) \((\alpha \in Q)\), \(g\) is \(Q\)-graded and we have the following identities for \(\mathfrak{h}\)-modules:

\[
g = n^+ \oplus \mathfrak{h} \oplus n^- = \bigoplus_{\alpha \in Q^+} g_{\pm \alpha}, \quad \mathfrak{h} = g_0.
\]

Hence the root system \(\Phi := \{\alpha \in Q \mid g_\alpha \neq \{0\}, \alpha \neq 0\}\) satisfies \(\Phi = \Phi^+ \cup \Phi^-\) where \(\Phi^\pm = \pm(\Phi \cap Q^+)\). The Weyl group \(W\) is a finite subgroup of \(\text{GL}(\mathfrak{h}^*)\) generated by the simple reflections \(s_i\) \((i \in I)\) acting via \(s_i(\alpha) = \alpha - \alpha(h_i)\alpha_i\) for all \(i \in I\), \(\alpha \in \mathfrak{h}^*\). We define

\[
\begin{align*}
\text{Aut}(\Phi) &= \{g \in \text{GL}(\mathfrak{h}^*) \mid g(\Phi) = \Phi\}, \\
\text{Aut}(A) &= \{\sigma : I \to I \text{ invertible} \mid a_{\sigma(i)\sigma(j)} = a_{ij} \text{ for all } i, j \in I\}.
\end{align*}
\]

Then \(\text{Aut}(\Phi) = W \rtimes \text{Aut}(A)\), with \(\text{Aut}(A)\) acting by relabelling.

We briefly review some important subgroups of

\[
\text{Aut}(g, \mathfrak{h}) = \{\sigma \in \text{Aut}(g) \mid |\sigma(\mathfrak{h}) = \mathfrak{h}\} \subset \text{Aut}(g).
\]

We have \(\text{Aut}(A) \subset \text{Aut}(g, \mathfrak{h})\) (acting by relabelling). Also, a braid group action on \(g\) is given by \(\text{Ad}(s_i) = \exp(\text{ad}(e_i)) \exp(\text{ad}(-f_i)) \exp(\text{ad}(e_i))\) for \(i \in I\). It extends the action of \(W\) on \(\mathfrak{h}\) dual to the one on \(\mathfrak{h}^*\) and satisfies \(\text{Ad}(W) \subset \text{Aut}(g, \mathfrak{h})\). The Chevalley involution \(\omega \in \text{Aut}(g, \mathfrak{h})\) is defined by swapping \(e_i\) and \(-f_i\) for all \(i \in I\); it commutes with \(\text{Ad}(W)\) and with \(\text{Aut}(A)\). Finally, the group \(\overline{H} := \text{Hom}(Q, C^*)\) naturally induces a subgroup \(\text{Ad}(\overline{H}) \subset \text{Aut}(g, \mathfrak{h})\) via \(\text{Ad}(\chi)|_{g_\alpha} = \chi(\alpha)\text{id}_{g_\alpha}\) for all \(\chi \in \overline{H}, \alpha \in Q\).

The elements of \(\text{Aut}(g, \mathfrak{h})\) can be dualized to elements of \(\text{Aut}(\mathfrak{H})\). Conversely, given \(g \in \text{Aut}(\Phi)\) there are \(\psi \in \text{Aut}(g, \mathfrak{h})\) whose restriction to \(\mathfrak{h}\) dualizes to \(g\). Indeed, from \(-\text{id}_{\mathfrak{h}^*} \in \text{Aut}(\Phi)\) and \(\text{Aut}(\Phi) = W \rtimes \text{Aut}(A)\), there exist unique \((w, \tau) \in W \times \text{Aut}(A)\) such that \(g = -w\tau\). Then \(\psi = \text{Ad}(w)\omega\tau \in \text{Aut}(g, \mathfrak{h})\) satisfies \((\psi|_{\mathfrak{h}})^* = g\).
2.1. Compatible decorations and involutions of $\Phi$. Given a subset $X \subseteq I$ denote the corresponding Cartan submatrix by $A_X = (a_{ij})_{i,j \in X}$ and consider the semisimple Lie algebra $\mathfrak{g}_X := \langle e_i, f_i, h_i \mid i \in X \rangle \subseteq \mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}_X = \mathfrak{h} \cap \mathfrak{g}_X$ and dual Weyl vector $\rho_X^\vee \in \mathfrak{h}_X$. The unique longest element $w_X$ of the Weyl group $W_X := \langle s_i \mid i \in X \rangle$ is an involution and there exists $\tau_{0,X} \in \text{Aut}(A_X)$ which satisfies

\begin{equation}
- w_X(\alpha_i) = \alpha_{\tau_{0,X}(i)} \quad \text{for all } i \in X.
\end{equation}

Note that $\text{Ad}(w_X)|_{\mathfrak{g}_X} = \tau_{0,X} \omega|_{\mathfrak{g}_X}$ and $\text{Ad}(w_X)^2|_{\mathfrak{h}_X} = \zeta(\alpha)\text{id}_{\mathfrak{h}_X}$ for all $\alpha \in \Phi$, where $\zeta \in \tilde{H}$ is defined by $\zeta(\alpha_i) := (-1)^{2\alpha_i(\rho_X)}$ for $i \in I$.

We can describe

\begin{align*}
\text{Aut}^{\text{inv}}(\mathfrak{g}, \mathfrak{h}) := \{ \psi \in \text{Aut}(\mathfrak{g}, \mathfrak{h}) \mid \psi^2 = \text{id}_{\mathfrak{h}} \}, \\
\text{Aut}^{\text{inv}}(\Phi) := \{ g \in \text{Aut}(\Phi) \mid g^2 = \text{id}_\Phi \}
\end{align*}

by means of combinatorial data. Define the set of compatible decorations as

\begin{equation}
\text{CDec}(A) = \{ (X, \tau) \mid X \subseteq I, \tau \in \text{Aut}(A), \tau^2 = \text{id}_I, \tau(X) = X, \tau|_X = \tau_{0,X} \}.
\end{equation}

In the associated Dynkin diagram one marks a decoration by filling the nodes corresponding to $X$ and drawing two-sided arrows for the nontrivial orbits of $\tau$.

**Example 2.1.** Let $A$ be of type $A_n$, $n \geq 2$. The compatible decorations are

\[
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \\
\bullet \quad \bullet \quad \bullet \quad \bullet
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\end{array}
\]

where $k \in \mathbb{Z}_{\geq 2}$, $p_1, p_k \in \mathbb{Z}_{\geq 0}$, $p_2, \ldots, p_{k-1} \in \mathbb{Z}_{\geq 1}$ and $r$, the number of $\tau$-orbits in $X$, is constrained by $0 \leq r \leq \lceil n/2 \rceil$.

Given $(X, \tau) \in \text{CDec}(A)$, we define

\begin{equation}
\theta = \theta(X, \tau) = -w_X \tau \in \text{Aut}^{\text{inv}}(\Phi).
\end{equation}

As explained above, the map dual to $\theta$, also given by $-w_X \tau$, can be extended to an element of $\text{Aut}^{\text{inv}}(\mathfrak{g}, \mathfrak{h})$ which we shall also call $\theta$. It is given by $\theta = \text{Ad}(w_X)\tau \omega$. As a consequence of properties of $\text{Ad}(w_X)$ mentioned earlier, we have

\begin{align}
\theta|_{\mathfrak{g}_X} &= \text{id}_{\mathfrak{g}_X}, \\
\theta^2|_{\mathfrak{h}_X} &= \zeta(\alpha)\text{id}_{\mathfrak{h}_X} \quad \text{for all } \alpha \in \Phi.
\end{align}

2.2. Generalized Satake diagrams and the restricted Weyl group. We choose a subset $I^* \subseteq I \setminus X$ such that it contains precisely one element from each $\tau$-orbit in $I \setminus X$. For $i \in I^*$ denote by $X(i) \subseteq X$ the union of connected components of $X$ neighbouring $\{i, \tau(i)\}$ and $\tilde{X}[i] := X(i) \cup \{i, \tau(i)\}$. By a minimal subdiagram of $(X, \tau) \in \text{CDec}(A)$ we mean any subdiagram of the form $\tilde{X}[i]$ for some $i \in I^*$. By definition $\tilde{X}[i]$ is a compatible decoration of $A_{\tilde{X}[i]}$; it is also known as a Satake diagram of (restricted) rank 1.

**Definition 2.2.** Generalized Satake diagrams are elements of the set

\[\text{GSat}(A) := \{ (X, \tau) \in \text{CDec}(A) \mid (X, \tau) \text{ contains no minimal subdiagram of the form } \bullet \rightarrow \bullet \}.\]

Hence, the compatible decorations in Example 2.1 are generalized Satake diagrams if and only if $p_1 = p_k = 0$ and $p_2 = \ldots = p_{k-1} = 1$.

**Remark 2.3.** Generalized Satake diagrams were first considered by Heck in [He84], who uses the symbol $\sigma$ to denote the negative of our map $\theta$. He also uses the term “Satake diagram” for any $(X, \tau)$ such that $X \subseteq I$, $\tau \in \text{Aut}(A)$, $\tau^2 = \text{id}_I$ and $\tau(X) = X$ (this properly contains the set $\text{CDec}(A)$) and the elements of $\text{GSat}(A)$ are called admissible Satake diagrams. However, typically the term “Satake diagram” denotes
those combinatorial data which classify involutions of $\mathfrak{g}$ up to conjugacy (and their fixed-point subalgebras), which is the reason for our nomenclature “compatible decoration” and “generalized Satake diagram”. 

Note that $(X, \tau)$ is a generalized Satake diagram precisely if
\begin{equation}
\forall (i, j) \in I \times X : \tau(i) = i, w_{X}(\alpha_{i}) = \alpha_{i} + \alpha_{j} \implies \alpha_{ij} \neq -1, \tag{2.9}
\end{equation}
which is the condition needed in [Ko14, Proof of Lemma 5.11, Step 1] and [BK16, Proof of Lemma 6.4]. One can show that (2.9) is equivalent to the following more compact conditions:
\begin{align*}
\forall i, j \in I : & \theta(\alpha_{i}) = -(\alpha_{i} + \alpha_{j}) \implies \alpha_{ij} \neq -1, \\
\forall i \in I : & (\theta(\alpha_{i})) (h_{i}) \neq -1.
\end{align*}

Satake diagrams can be defined as the following subset of compatible decorations of $A$:
\begin{equation}
\text{Sat}(A) = \{(X, \tau) \in \text{CDec}(A) \mid \forall i \in I \setminus X : i = \tau(i) \implies \zeta(\alpha_{i}) = 1\}. \tag{2.10}
\end{equation}
Satake diagrams classify involutive Lie algebra automorphisms up to conjugacy, see e.g. [Ar62]. More precisely, in our notation, for $(X, \tau) \in \text{Sat}(A)$ and $\gamma \in (\mathbb{C}^{\times})^{I^{*}}$ define $\chi_{\gamma} \in \tilde{H}$ by
\begin{equation}
\chi_{\gamma}(\alpha_{i}) = \begin{cases} 
1 & \text{if } i \in X, \\
\gamma_{i} & \text{if } i \in I^{*}, \\
\gamma_{\tau(i)}(\alpha_{i}) & \text{if } i \in (I \setminus X) \setminus I^{*},
\end{cases} \tag{2.11}
\end{equation}
cf. [BK16, Eqs. (5.1-5.2)]. Then it follows from (2.8) that $\theta_{\gamma} := \text{Ad}(\chi_{\gamma}) \theta$ satisfies $(\theta_{\gamma})^{2} = \text{id}_{\mathfrak{g}}$.

If $(X, \tau) \in \text{CDec}(A) \setminus \text{GSat}(A)$ then there exist $i \in I \setminus X$, $j \in X$ such that $\tau(i) = i$, $\alpha_{ji} = -1$ and $X(i) = \{ j \}$. Hence $\rho_{X}^{\gamma} = \frac{1}{\gamma_{j}} h_{j}$ so that $\zeta(\alpha_{i}) = (-1)^{n_{\nu}} = -1$ implying $(X, \tau) \notin \text{Sat}(A)$. Consequently $\text{Sat}(A) \subseteq \text{GSat}(A)$. The complement $\text{GSat}(A) \setminus \text{Sat}(A)$ is empty if and only if $A$ is of type $A_{n}$. We refer the reader to the classification in [He84, Table I]. Since this does not distinguish between elements of $\text{Sat}(A)$ and $\text{GSat}(A) \setminus \text{Sat}(A)$, for later convenience we list the elements of $\text{GSat}(A) \setminus \text{Sat}(A)$, see Table 1.

**Table 1.** All elements of $\text{GSat}(A) \setminus \text{Sat}(A)$ for indecomposable Cartan matrices $A$. By a case-by-case analysis there is a unique $i \in I$ such that $i = \tau(i)$ and $\zeta(\alpha_{i}) = -1$; we have indicated the corresponding node in the diagrams. The classical diagrams are labelled in the usual way. For types $C_{n}$ and $D_{n}$ upper bounds on $i$ are imposed to avoid the cases when $\theta$ is an involution whose fixed-point subalgebra is isomorphic to $\mathfrak{g}l_{n}$.

Consider the real vector space $V = \mathbb{R} \Phi$. For fixed $\theta \in \text{Aut}^{\text{inv}}(\Phi)$ we have the decomposition $V = V^{\theta} \oplus V^{-\theta}$. Denote by $\pi : V \to V$ the corresponding projection onto $V^{-\theta}$. The restricted roots are the elements of $\Phi = \{ \alpha \in \Phi \} \setminus \{ 0 \}$. Given $\theta \in \text{Aut}^{\text{inv}}(\Phi)$, $\Phi$ is not always a root system in its own right. According to [He84, Thm. 6.1], $\Phi$ is a (possibly non-reduced or empty) root system precisely if $\theta = \theta(X, \tau) = -w_{X} \tau$, where $(X, \tau) \in \text{GSat}(A)$ or $(X, \tau) = \bullet$. 

If \( \theta = \theta(X, \tau) \) with \((X, \tau) \in \text{CDec}(A)\) it follows straightforwardly that \( W_X \) is a normal subgroup of
\[
W^\theta = \{ w \in W \mid \theta w = w \theta \} = \{ w \in W \mid w = w_X \tau(w)w_X \}.
\]

Now consider the restricted Weyl group
\[
\overline{W} = \{ w|_{V^{-\theta}} \mid w \in W, w(V^{-\theta}) \subseteq V^{-\theta} \}.
\]

By [He84, Prop. 3.1] we have \( \overline{W} \cong W^\theta/W_X \). For \( i \in I^* \) denote \( X[i] = X \cup \{ i, \tau(i) \} \) and let \( \pi_i \in \text{GL}(V^{-\theta}) \) be the element that sends \( \pi_i \) to \(-\pi_i\) and fixes all \( \beta \in V^{-\theta} \) with \( \beta(h_i) = 0 \).

**Theorem 2.4** ([He84] and [Lu76]). Let \((X, \tau) \in \text{CDec}(A)\). The following conditions are equivalent:

(i) We have \((X, \tau) \in \text{GSat}(A)\).

(ii) For all \( i \in I^* \), \( \overline{\pi}_i \in \overline{W} \).

(iii) For all \( i \in I^* \), \( \overline{\pi}_i := w_Xw_X[i] \) lies in \( W^\theta \) and satisfies \( \overline{\pi}_i|_{V^{-\theta}} = \overline{\pi}_i \).

(iv) For all \( i \in I^* \), \( \overline{\tau}_{0,X[i]} \) preserves \( X \).

(v) The restricted Weyl group \( \overline{W} \) is the Weyl group of \( \overline{\pi} \).

(vi) The set \( \{ \overline{\pi}_i \mid i \in I^* \} \) is a Coxeter system for the group it generates.

**Proof.** The equivalence of the first five statements is shown in [He84, Lemma 3.2, Thm. 3.3, Thm. 4.4]. The implication \((iv) \implies (vi)\) is shown in [Lu76, 5.9 (i)] (also see [Lu02, 25.1]). Finally, its converse follows from the observation that if condition \((iv)\) fails then for some \( i \in I^* \), \( w_X[i] \) and \( w_X \) do not commute so that \( \overline{\pi}_i^2 \neq \text{id}_V \). \( \square \)

3. The subalgebra \( \mathfrak{k} \)

For \((X, \tau) \in \text{Sat}(A)\) and a suitable choice of \( \gamma \in (\mathbb{C}^*)^{I^*} \) the \( \theta_\gamma \)-fixed subalgebra \( \mathfrak{k} \) of \( \mathfrak{g} \) can be presented in terms of generators; see e.g. [Ko14, Lemma 2.8] in the case that all \( \gamma_i = 1 \). This motivates the following seemingly ad hoc definition, where we permit a more general \( \gamma \).

**Definition 3.1.** For \((X, \tau) \in \text{CDec}(A)\) and \( \gamma \in (\mathbb{C}^*)^{I \setminus X} \) define \( \mathfrak{k}_\gamma = \mathfrak{k}_\gamma(X, \tau) \) to be the Lie subalgebra of \( \mathfrak{g} \) generated by \( \mathfrak{g}_X, \mathfrak{h}^\theta \) and
\[
b_{i, \gamma} = f_i + \gamma_i \theta(f_i) \quad \text{for all } i \in I \setminus X.
\]

It is convenient to suppress the dependence on \( \gamma \) and simply write \( b_i \) and \( \mathfrak{k} \) if there is no cause for confusion. We denote \( b_i = f_i \) if \( i \in X \). Since \( \mathfrak{h}_X \subseteq \mathfrak{h}^\theta \) it follows that \( \mathfrak{k} \) is generated by \( \mathfrak{n}_X^\perp := \{ e_i \mid i \in X \} \), \( \mathfrak{h}^\theta \) and \( b_i \) for \( i \in I \). Owing to (2.1-2.2) and (2.7), these satisfy
\[
[e_i, b_j] = \delta_{ij}h_i \in \mathfrak{h}^\theta \quad \text{for all } i \in X, \ j \in I,
\]
\[
[h, b_j] = -\alpha_j(h)b_j \quad \text{for all } h \in \mathfrak{h}^\theta, \ j \in I,
\]
\[
[h, e_j] = \alpha_j(h)e_j \quad \text{for all } h \in \mathfrak{h}^\theta, \ j \in X,
\]
\[
[h, h'] = 0 \quad \text{for all } h, h' \in \mathfrak{h}^\theta,
\]
\[
\text{ad}(e_i)^{1-a_{ij}}(e_j) = 0 \quad \text{for all } i, j \in X, \ i \neq j.
\]
By setting \( m = 1 - \alpha_{ij} \) in Lemmas (A.1-A.3) one also obtains analogues of Serre relations among the generators \( b_i \). Namely, for \( i, j \in I \) such that \( i \neq j \),
\[
\begin{align*}
\text{ad}(b_i)^{-1-a_{ij}}(b_j) &= \\
&= \begin{cases}
(1 + \zeta_{i,j}) [\theta(f_i), [f_i, f_j]] \in \mathfrak{n}_X^- & \text{if } \theta(\alpha_i) + \alpha_i + \alpha_j \in \Phi^- \setminus \Phi^+, \alpha_{ij} = -1, \\
-18\gamma_i^2 c_j & \text{if } \theta(\alpha_i) + \alpha_i + \alpha_j = 0, \alpha_{ij} = -3, \\
-\gamma_i (2h_i + h_j) & \text{if } \theta(\alpha_i) + \alpha_i + \alpha_j = 0, \alpha_{ij} = -1, \\
\{\gamma_i + \zeta(\alpha_i)\gamma_j\} [\theta(f_i), f_j] \in \mathfrak{n}_X^+ & \text{if } \theta(\alpha_i) + \alpha_i + \alpha_j \in \Phi^-, \alpha_{ij} = 0, \\
\gamma_i h_i - \alpha_i h_j & \text{if } \theta(\alpha_i) + \alpha_j = 0, \alpha_{ij} = 0, \\
2(\gamma_i + \gamma_j) b_i & \text{if } \theta(\alpha_i) + \alpha_j = 0, \alpha_{ij} = -1, \\
-\gamma_i b_j & \text{if } \theta(\alpha_i) + \alpha_i = 0, j \in I \setminus X, \alpha_{ij} = -1, \\
-3\gamma_i [b_i, b_j] & \text{if } \theta(\alpha_i) + \alpha_i = 0, j \in I \setminus X, \alpha_{ij} = -2, \\
-6\gamma_i^2 b_j - 3\gamma_i [b_i, [b_i, b_j]] & \text{if } \theta(\alpha_i) + \alpha_i = 0, j \in I \setminus X, \alpha_{ij} = -3, \\
0 & \text{otherwise}.
\end{cases}
\tag{3.7}
\end{align*}
\]

\section{Basic structure of \( \mathfrak{t} \).}

In order to state the main result of this section, we need some notation. For all \( i, j \in I \) such that \( i \neq j \) we write \( \lambda_{ij} := (1 - \alpha_{ij}) \alpha_i + \alpha_j \in Q^+ \setminus \Phi^+ \). Consider the sets
\[
\begin{align*}
I_{\text{diff}} &= \{ i \in I \mid i \neq \tau(i) \text{ and } (\theta(\alpha_i))(h_i) \neq 0 \} = \{ i \in I \mid i \neq \tau(i) \text{ and } \exists j \in X[i] \text{ s.t. } \alpha_{ij} < 0 \} \\
\Gamma &= \Gamma(X, \tau) = \{ \gamma \in (C^\times)^{I \setminus X} \mid \forall i \in I^* : \gamma_i \neq \gamma_{\tau(i)} \implies i \in I_{\text{diff}} \}.
\end{align*}
\]

For \( i \in I^\ell \) with \( \ell \in \mathbb{Z}_{>0} \) we write \( \alpha_i = \sum_{r=1}^\ell \alpha_{ir} \) and
\[
b_i = \text{ad}(b_{i_1}) \cdots \text{ad}(b_{i_{\ell-1}})(b_{i_\ell}), \quad e_i = \text{ad}(e_{i_1}) \cdots \text{ad}(e_{i_{\ell-1}})(e_{i_\ell}), \quad f_i = \text{ad}(f_{i_1}) \cdots \text{ad}(f_{i_{\ell-1}})(f_{i_\ell}).
\]

Observe that \( n^- = \text{Sp}\{ f_i \mid i \in I^\ell, \ell > 0 \} \) and \( n^+_X = \text{Sp}\{ e_i \mid i \in X^\ell, \ell > 0 \} \). Hence for all \( \ell \in \mathbb{Z}_{>0} \) we can choose \( J_\ell \subseteq I^\ell \) such that \( \{ f_i \}_{i \in J_\ell} \) is a basis for \( \text{Sp}\{ f_i \}_{i \in I^\ell} \) and \( \{ e_i \}_{i \in J_{X, \ell}} \) is a basis for \( \text{Sp}\{ e_i \}_{i \in X^\ell} \) where \( J_{X, \ell} := J_\ell \cap X^\ell \). Then \( \{ f_i \}_{i \in J} \) with \( J := \bigcup_{\ell \in \mathbb{Z}_{>0}} J_\ell \) is a basis of \( n^- \) and \( \{ e_i \}_{i \in J_X} \) with \( J_X := \bigcup_{\ell \in \mathbb{Z}_{>0}} J_{X, \ell} \) is a basis of \( n^+_X \).

\begin{theorem}
Let \( (X, \tau) \in \text{CDec}(A) \) and \( \gamma \in (C^\times)^{I \setminus X} \). The following statements are equivalent:
\begin{enumerate}
\item We have \( (X, \tau) \in \text{GSat}(A) \) and \( \gamma \in \Gamma \).
\item For all \( i, j \in I \) such that \( i \neq j \) we have
\[
\text{ad}(b_i)^{-1-a_{ij}}(b_j) \in \mathfrak{n}_X^+ + \mathfrak{h}^0 \oplus \bigoplus_{\ell \in I^\ell, \alpha_{ir} < \lambda_{ij}} \mathbb{C} b_{i_r}.
\tag{3.8}
\]
\item We have the following identity for \( \mathfrak{h}^0 \)-modules:
\[
\mathfrak{t} = \mathfrak{n}_X^+ + \mathfrak{h}^0 \oplus \bigoplus_{i \in J} \mathbb{C} b_i.
\tag{3.9}
\]
\item We have
\[
\mathfrak{t} \cap \mathfrak{k} = \mathfrak{h}^0.
\tag{3.10}
\]
\end{enumerate}
\end{theorem}

In the fixed-point case \( \mathfrak{t} = \mathfrak{g}^{0,\gamma} \) (3.10) is trivially satisfied (note that \( \mathfrak{h}^0 = \mathfrak{h}^{0,\gamma} \)).

\begin{proof}[Proof of Theorem 3.2]
\begin{enumerate}
\item \( \iff \) (iii): This is a direct consequence of (3.7).
\item \implies (iii): Owing to (3.3-3.5) it is sufficient to prove (3.15) as an identity for vector spaces. First we prove that \( \mathfrak{t} = \mathfrak{n}_X^+ + \mathfrak{h}^0 + \text{Sp}\{ b_i \mid i \in J \} \). From (3.2-3.3) it follows that
\[
\mathfrak{t} = \mathfrak{n}_X^+ + \mathfrak{h}^0 + \langle b_j \rangle_{j \in I} = \mathfrak{n}_X^+ + \mathfrak{h}^0 + \sum_{\ell \in \mathbb{Z}_{>0}} \sum_{i \in I^\ell} \mathbb{C} b_i
\tag{3.11}
\]
\end{enumerate}
\end{proof}
as vector spaces. Hence it suffices to prove that for all \( j \in \bigcup \ell I^\ell \) we have

\[
(3.12) \quad b_j \in n_X^+ + b^\theta + \text{Sp}\{\alpha | i \in J\}.
\]

We will prove this by induction with respect to the height \( \ell \). Since for all \( j \in I \) we have \( \dim(g_{-\alpha}) = 1 \) and hence \( (j) \in J \), the case \( \ell = 1 \) is trivial. Now fix \( \ell \in \mathbb{Z}_{>1} \) and assume that (3.12) holds true for all smaller positive integers. Fix \( j \in I^\ell \) and repeatedly apply the Serre relations (2.2) to obtain that for all \( i \in J_\ell \) there exist \( a_i \in \mathbb{C} \) such that \( f_j = \sum_{i \in J_\ell} a_i f_i \). Hence, by virtue of (ii) and equations (3.2-3.3) it follows that

\[
b_j - \sum_{i \in J_\ell} a_i b_i \in n_X^+ + b^\theta + \text{Sp}\{\alpha | i \in J, \alpha_i < \alpha_j\}.
\]

Using the induction hypothesis for the elements \( b_i \) in the last summation one obtains (3.12).

It remains to show that the sum in (3.12) is direct. Let \( j \in J \). Then \( f_j \) is nonzero. Because of the explicit formula (3.1) we have

\[
(3.13) \quad b_j - f_j \in n_X^+ + b^\theta + \text{Sp}\{\alpha | i \in J, \alpha_i < \alpha_j\}.
\]

Hence \( f_j = \pi_{-\alpha_j}(b_j) \) for all \( j \in J \), where \( \pi_{-\alpha_j} \) is the projection on \( g_\alpha \) for \( \alpha \in \Phi \), see (2.3). Thus the linear independence of \( \{f_j\}_{j \in J} \) together with (2.3) implies that the sum is direct.

\( (iii) \implies (iv) \): By definition, \( b^\theta \subseteq \ell \cap h \) so it suffices to show that \( \ell \cap h \subseteq b^\theta \). Suppose \( h \in \ell \cap b^\theta \). Since \( \pi_{-\alpha_j}(b_j) = f_j \) and the triangular decomposition (2.3), part (iii) implies \( h \in n_X^+ + b^\theta \subseteq g^\theta \) so \( h \in b^\theta \).

\( (iv) \implies (ii) \): We prove the contrapositive. If (3.8) fails then (3.14) and (3.7) imply that either \( \gamma_j h_i - \gamma_j h_j \in \ell \cap (h \backslash b^\theta) \) with \( \gamma_i = \gamma_j \) or \( 2h_i + h_j \in \ell \cap (h \backslash b^\theta) \). In either case (3.10) does not hold.

Given \( i \in I \), by applying \( \theta \) to \( \theta(h_i) - h_i - \theta(h_{\tau(i)}) + h_{\tau(i)} \in g_X \cap h \) we obtain that \( h_i - h_{\tau(i)} \) is fixed by \( \theta \). As a consequence,

\[
(3.14) \quad b^\theta = \bigoplus_{i \in X} \mathbb{C} h_i \oplus \bigoplus_{i \notin \tau(i)} \mathbb{C}(h_i - h_{\tau(i)})
\]

so that \( \dim(b^\theta) = |I| - |I^*| \). Hence, given \( (X, \tau) \in \text{GSat}(A), \gamma \in \Gamma \) and \( J \) as specified before Theorem 3.2, by (3.9) we obtain a standard basis for \( \ell \):

\[
(3.15) \quad \{e_i | i \in J_X\} \cup \{h_i | i \in X\} \cup \{h_i - h_{\tau(i)} | i \in I^*, i \neq \tau(i)\} \cup \{b_i | i \in J\}.
\]

We denote \( \Phi_X = \Phi \cap Q_X \). Since \( |J| = |\Phi|/2 \), (3.15) implies

\[
(3.16) \quad \dim(\ell) = |\Phi_X|/2 + |I| - |I^*| + |\Phi|/2.
\]

**Corollary 3.3.** Let \((X, \tau) \in \text{GSat}(A)\) and \( \gamma \in \Gamma \). The generators \( h_i, e_i (i \in X), h_i - h_{\tau(i)} (i \in I^*, i \neq \tau(i)), b_i (i \in I) \) and the relations (3.2-3.7) provide a presentation for \( \ell \).

**Proof.** There are no relations for the \( b_i \) other than (3.2), (3.3) and (3.7): otherwise applying \( \pi_{-\alpha} \) with \( \alpha \in \Phi^+ \) maximal produces a relation for the \( f_i \) not given by (2.1) or (2.2). \( \square \)

### 3.2. Semidirect product decompositions of \( \ell \)

In this section we assume that \( A \) is indecomposable, so that \( g \) is simple. In order to describe the derived subalgebra of \( \ell \) recall the set \( I_{\text{diff}} \subset I^* \) and define

\[
(3.17) \quad I_{\text{ns}} = \{i \in I | \theta(\alpha(i))(h_i) = -2\} = \{i \in I | i = \tau(i), \bar{X}(i) = 0\},
\]

\[
I_{\text{nst}} = \{j \in I_{\text{ns}} | \forall i \in I_{\text{ns}} \alpha_{ij} \in 2\mathbb{Z}\}.
\]

**Proposition 3.4.** Let \((X, \tau) \in \text{GSat}(A)\) and \( \gamma \in \Gamma \). The set

\[
\{e_i | i \in J_X\} \cup \{h_i | i \in X\} \cup \{h_i - h_{\tau(i)} | i \in I^* \setminus I_{\text{diff}}, i \neq \tau(i)\} \cup \{b_i | i \in J, i \neq (j) \text{ with } j \in I_{\text{nst}}\}
\]

forms a basis for the derived subalgebra \( \mathfrak{g} \) and we have

\[
(3.18) \quad \ell = \ell' \times \bigoplus_{i \in I_{\text{nst}}} \mathbb{C}(h_i - h_{\tau(i)}) \oplus \bigoplus_{j \in I_{\text{nst}}} \mathbb{C} b_j.
\]
Proof. Fix $(X, \tau) \in \text{GSat}(A)$. Note that neither $h_i - h_{\tau(i)} (i \in I_{\text{diff}})$ nor $b_j (j \in I_{\text{nsf}})$ is a linear combination of Lie brackets in $\mathfrak{k}$. This follows from Corollary 3.3 and (3.2-3.7): these elements do not appear as in the expressions for Lie brackets in the defining relations of $\mathfrak{k}$.

It now suffices to show that the remaining basis elements specified in (3.15) are linear combinations of Lie brackets in $\mathfrak{k}$, for which we argue as follows.

- For $b_i$ with $i \in J$ and $e_i$ with $i \in J_{X, \ell}$ with $\ell > 1$, this holds by definition.
- For $e_i, f_i, h_i$ with $i \in X$, this follows from (3.2-3.4).
- For $h_i - h_{\tau(i)}$ with $i \in I^* \setminus I_{\text{diff}}$ and $i \neq \tau(i)$, the given condition is equivalent to $w_X(\alpha_i) = \alpha_i$ and $a_{i, \tau(i)} = 0$. Hence (3.7) implies that $h_i - h_{\tau(i)} = \gamma_i^{-1} [b_i, b_{\tau(i)}]$.
- For $b_j$ with $X(j) \neq \emptyset$ there exists $i \in X$ such that $a_{ij} \neq 0$. By (3.3) we have $b_j = -a_{ij}^{-1} [h_i, b_j]$.
- For $b_j$ with $j \neq \tau(j)$, by (3.3) we have $b_j = (a_{\tau(j), j} - 2)^{-1} [h_j - h_{\tau(j)}, b_j]$.
- For $b_j$ with $j \neq \tau(j)$, by (3.3) we have $b_j = (a_{\tau(j), j} - 2)^{-1} [h_j - h_{\tau(j)}, b_j]$.
- For $b_j$ with $j \neq \tau(j)$, by (3.3) we have $b_j = (a_{\tau(j), j} - 2)^{-1} [h_j - h_{\tau(j)}, b_j]$.
- For $b_j$ with $j \neq \tau(j)$, by (3.3) we have $b_j = (a_{\tau(j), j} - 2)^{-1} [h_j - h_{\tau(j)}, b_j]$.

It follows that the codimension of $\mathfrak{t}'$ in $\mathfrak{k}$ equals $|I_{\text{diff}}| + |I_{\text{nsf}}|$. For $(X, \tau) \in \text{Sat}(A)$, in [Le02, Sec. 7, Variation 1] it was noted that $|I_{\text{diff}}| \leq 1$ if $A$ is of finite type. In light of the above it is natural to generalize this: we can include the set $I_{\text{nsf}}$ and allow $(X, \tau) \in \text{GSat}(A)$. Then still we have $|I_{\text{diff}}| + |I_{\text{nsf}}| \geq 1$. There are generalized Satake diagrams with $|I_{\text{diff}}| + |I_{\text{nsf}}| = 1$ unless $A$ is of type $E_8$, $F_4$ or $G_2$. From Table 1 it follows that the only elements of $\text{GSat}(A) \setminus \text{Sat}(A)$ for which $|I_{\text{diff}}| + |I_{\text{nsf}}| = 1$ are of the form $\begin{array}{c} 1 \\ \circ \\ \circ \\ \circ \\ \circ \end{array}$ with $n > 2$ in which case $I_{\text{nsf}} = \{1\}$ and $\zeta(\alpha_2) = -1$.

Remark 3.5. From a case-by-case analysis of Satake diagrams and the associated fixed-point subalgebras (e.g. see [Ar62]) one sees that precisely when $|I_{\text{diff}}| + |I_{\text{nsf}}| = 1$ the subalgebra $\mathfrak{g}^0$ is reductive. Its centre is generated by a linear combination of either $h_i - h_{\tau(i)} (i \in I_{\text{diff}})$ or $b_i (i \in I_{\text{nsf}})$ and at least one other standard basis element of $\mathfrak{t}$.

Definition 3.6. The set of weak Satake diagrams is

$$\text{WSat}(A) = \{ (X, \tau) \in \text{GSat}(A) \setminus \text{Sat}(A) \mid (X, \tau) \neq \begin{array}{c} 1 \\ \circ \\ \circ \\ \circ \\ \circ \end{array} \}.$$ 

For $(X, \tau) \in \text{WSat}(A)$ we will obtain a semidirect product decomposition in terms of a reductive Lie subalgebra and a nilpotent ideal. For any $r \in \mathbb{Z}_{\geq 0}$ and any $i \in I$ denote by $\mathfrak{t}(i)_r$ the span of all $b_j$ such that the coefficient of $\alpha_j$ in $\alpha_j$ is precisely $r$. Consider the subspace

$$\mathfrak{t}(i) := \bigoplus_{r=1}^{\infty} \mathfrak{t}(i)_r.$$ 

Proposition 3.7. Let $(X, \tau) \in \text{WSat}(A)$, $\gamma \in \Gamma$ and $i$ the unique element of $I \setminus X$ such that $i = \tau(i)$ and $\zeta(\alpha_i) = -1$. Then $\mathfrak{t}(i)$ is nilpotent of class 2: $\mathfrak{t}(i)_r = \{0\}$ if $r > 2$ and we have the lower central series

$$\mathfrak{t}(i) = \mathfrak{t}(i)_1 \oplus \mathfrak{t}(i)_2 \supset \mathfrak{t}(i)_2 \supset \{0\}.$$ 

Moreover, we write $\mathfrak{t}_i := \mathfrak{t} \cap \mathfrak{g}_i \setminus \{i\}$ and $\theta_i$ for the restriction of $\theta$ to $\mathfrak{g}_i \setminus \{i\}$; then $\mathfrak{t}_i$ is the fixed-point subalgebra of $\theta_i$, both $\mathfrak{t}(i)_1$ and $\mathfrak{t}(i)_2$ are $\mathfrak{t}_i$-modules under the adjoint action, $\mathfrak{t}(i)$ is an ideal of $\mathfrak{t}$ and $\mathfrak{t} = \mathfrak{t}(i) \times \mathfrak{t}_i$.

Proof. Note that (3.7) implies, for all $j \in I \setminus \{i\}$, that

$$\text{(3.19)} \quad \text{ad}(b_j)^{1-a_{ij}}(b_j) = 0$$

$$\text{(3.20)} \quad \text{ad}(b_j)^{1-a_{ij}}(b_j) \in \bigoplus_{r=1}^{\infty} \zeta(\alpha_j)^r \mathfrak{g}(b_j)^r(b_j) \subseteq \mathfrak{t}(i)_1.$$ 

Since (3.3) and (3.20) are the only relations in $\mathfrak{t}$ with $b_i$ appearing on the right-hand side, it follows that $\mathfrak{t}_i = (\mathfrak{u}^X_i, \mathfrak{h}^0, \mathfrak{t}(i)_0)$ and $\mathfrak{t} = \mathfrak{t}(i) \oplus \mathfrak{t}_i$ (as vector spaces). Deleting the node $i$ from any diagram in Table 1 one
yields that the coefficient of \( \alpha_i \). Suppose there exists \( \phi \in U_n(g) \) implies that \( \tau(i) \) is an ideal. Automatically we have that \( \phi(x) = x \) for all \( s \in \mathbb{Z}_{\geq 1} \). A case-by-case analysis using Table 1 yields that the coefficient in front of \( \alpha_i \) in the highest root of \( \Phi \) is always 2. This implies \( \tau(i) = 0 \) if \( r > 2 \) and we obtain the indicated lower central series.

\[ \text{(3.21)} \]

Example 3.8. We discuss two examples of \( (X, \tau) \in GSat(A) \setminus Sat(A) \).

(i) The smallest such \( \mathfrak{t} \) occurs when \( (X, \tau) = \frac{1}{2} \mathfrak{g} \mathfrak{g} \). By definition, \( \mathfrak{t} \) is the subalgebra of \( \mathfrak{g} \mathfrak{p}_4 \) generated by \( b_1 = f_1 + \gamma_1 \theta(f_1) \) for some \( \gamma_1 \in \mathbb{C}^X \) and \( b_2 = f_2, e_2, h_2 \). The relations (3.2-3.7) specialize to

\[ \begin{align*}
[e_2, b_1] &= 0, & [e_2, b_2] &= h_2, & [h_2, b_1] &= b_1, & [h_2, b_2] &= -2b_2, & [h_2, e_2] &= 2e_2, \\
[b_1, [b_1, [b_1, b_2]]] &= 0, & [b_2, [b_2, b_2]] &= 0.
\end{align*} \]

\[ \text{(3.21)} \]

According to (3.15), a standard basis of \( \mathfrak{t} \) is given by \( \{e_2, h_2, b_1, b_2, b(1,2), b(1,1,2)\} \). Proposition 3.4 implies \( \mathfrak{t} = \mathfrak{t}' \) and Proposition 3.7 yields the nontrivial Levi decomposition \( \mathfrak{t} = \mathfrak{g} \mathfrak{p}(b_1, b(1,2), b(1,1,2)) \times \mathfrak{sp}(e_2, h_2, b_2) \) with the radical isomorphic to the 3-dimensional Heisenberg Lie algebra and the Levi subalgebra isomorphic to \( \mathfrak{sl}_2 \). In particular it follows from (3.21) that \( b(1,1,2) \) is central.

(ii) Proposition 3.7 excludes the generalized Satake diagram \( (X, \tau) = \frac{1}{2} \mathfrak{g} \mathfrak{g} \). It is the only element of \( GSat(A) \setminus Sat(A) \) such that \( \mathfrak{t} \) is a reductive Lie algebra as we will see now. By definition, \( \mathfrak{t} \) is the subalgebra of \( \mathfrak{g} = \mathfrak{g}(G_2) \) generated by \( e_1, h_1, b_1 = f_1 \) and \( b_2 = f_2 + \gamma_2 \theta(f_2) \) for some \( \gamma_2 \in \mathbb{C}^X \). The relations (3.2-3.7) give

\[ \begin{align*}
[e_1, b_1] &= h_1, & [e_1, b_2] &= 0, & [h_1, b_1] &= -2b_1, & [h_1, b_2] &= b_2, & [h_1, e_1] &= 2e_1, \\
[b_1, [b_1, b_2]] &= 0, & [b_2, [b_2, b_2]] &= 0.
\end{align*} \]

\[ \text{(3.22)} \]

A standard basis of \( \mathfrak{t} \) is given by \( \{e_1, h_1, b_1, b_2, b(2,2), b(2,2,2), b(2,2,2,2)\} \). Proposition 3.4 yields \( \mathfrak{t} = \mathfrak{t}' \). Moreover, using (3.22), the adjoint action of \( e_1, b_1 \) and \( b_2 \) on \( \mathfrak{t} \) implies that any ideal of \( \mathfrak{t} \) equals \( \mathfrak{t} \) if it contains any of the above standard basis elements. Then some straightforward computations show that \( \mathfrak{t} \) is in fact a simple Lie algebra and hence isomorphic to \( \mathfrak{sl}_3 \).

Proposition 3.9. Let \( (X, \tau) \in GSat(A) \setminus Sat(A) \) and \( \gamma \in \Gamma \). Let \( \phi \in Aut(g) \) be such that 1 is a simple root of the minimal polynomial of \( \phi \). Then \( \mathfrak{t} \) is not the fixed-point subalgebra of \( \phi \).

Hence \( \mathfrak{t} \) is not the fixed-point subalgebra of any semisimple (in particular, finite-order) automorphism of \( g \). Nevertheless, in Section 4 we will show that the subalgebra \( \mathfrak{t} = (X, \tau) \) can be quantized resulting in a coideal subalgebra possessing a universal K-matrix if \( (X, \tau) \in GSat(A) \).

Proof of Proposition 3.9. We first show this for the case when \( (X, \tau) = \mathfrak{g} \mathfrak{p} \). Suppose there exists \( \phi \in Aut(g) \) such that \( \mathfrak{t} = \mathfrak{g} \mathfrak{p} \). From \( [b_2, b_1] = 3b_1 \) and \( [h_2, e_1] = -3e_1 \) one establishes straightforwardly that \( \phi(h_2) \in \mathfrak{h} \) and hence that \( \phi(h_2) = \frac{1}{2}(m - 1)h_1 + mh_2 \) for some \( m \in \mathbb{C} \). Next, from \( \theta(f_2) \in \mathfrak{g}_{\alpha_1 + \alpha_2} \) it follows that \( [h_2, b_2] = -2b_2 \); hence \( \phi(f_2) = m.f_2 + \frac{1}{2}(1 - m)b_2 \). Combining this with \( [f_2, b_2] \in \mathfrak{n}_X^+ \) one obtains \( m = 1 \). But this means that \( h_2 \) and \( f_2 \) are also fixed points of \( \phi \), contrary to assumption. Hence such \( \phi \) does not exist.

Now let \( (X, \tau) \in WSat(A) \). Since \( \mathfrak{t} \) has a nonabelian nilpotent ideal by Proposition 3.7, \( \mathfrak{t} \) is not a reductive Lie algebra. Hence [Jac62, Thm. 1] implies the desired conclusion.

Finally we comment on the centre \( \mathfrak{z} \) of \( \mathfrak{t} \) for \( (X, \tau) \in WSat(A) \). In Example 3.8 (i) we saw that it is one-dimensional if \( (X, \tau) = \mathfrak{g} \mathfrak{p} \). Let \( \mathfrak{c} \in \mathfrak{z} \) and as before denote by \( i \) the unique element of \( I \setminus X \) such that \( i = \tau(i) \) and \( \zeta(\alpha_i) = -1 \). Proposition 3.7 implies that \( c = c' + c'' \) with \( c' \in \mathfrak{t} \) and \( c'' \in \mathfrak{t}(i) \) and for all \( x \in \mathfrak{t} \) and \( y \in \mathfrak{t}(i) \) we have \( [x, c'] = 0 \) and \( [x, c''] + [y, c'] + [y, c''] = 0 \). From \( [x, c'] = 0 \) it follows that, unless \( c' = 0 \),
(X, τ) must be the diagram \( \xrightarrow{1} \xrightarrow{2} \xrightarrow{\cdots} \xrightarrow{n} \) and \( c' \in \mathbb{C}b_1 \); note that in this case \([b_1, b_1] \neq 0\). Therefore \( c' = 0 \) and hence \( c \in \mathfrak{t}(i) \). Since the centre of \( \mathfrak{t}(i) \) is \( \mathfrak{t}(i)_2 \) we must have \( \mathfrak{z} \subseteq \mathfrak{t}(i)_2 \). Define
\[ \mathcal{J}_{\text{even}} := \{ j \in \mathcal{J} | \forall k \in \mathcal{I} \setminus X \text{ the coefficient of } \alpha_j \text{ in front of } \alpha_k \text{ is even} \}. \]

**Conjecture 3.10.** Let \((X, \tau) \in \text{WSat}(A)\). Then \( \mathfrak{z} \) is generated by a single element of \( \mathfrak{t}(i)_2 : = \bigoplus_{j \in \mathcal{J}_{\text{even}}} \mathbb{C}b_j \subset \mathfrak{t}(i)_2 \).

3.3. **The universal enveloping algebra** \( U(\mathfrak{t}) \). Let \((X, \tau) \in \text{GSat}(A)\) and \( \gamma \in \Gamma \). We identify \( \mathfrak{t} \) with its image in \( U(\mathfrak{t}) \) under the canonical Lie algebra embedding. The generators of \( U(\mathfrak{t}) \) corresponding to \( b_i \) \((i \in \mathcal{I} \setminus X)\) can be modified by scalar terms, which is a straightforward generalization of [Ko14, Cor. 2.9].

**Proposition 3.11.** For \((X, \tau) \in \text{GSat}(A), \gamma \in \Gamma \) and \( \sigma \in \mathbb{C}^{I \setminus X} \), the universal enveloping algebra \( U(\mathfrak{t}_\gamma)^\sigma \) is generated by \( e_i, f_i \ (i \in X), h \in \mathfrak{h}^0 \) and
\[
(3.23) \quad b_i; \gamma, \sigma = f_i + \gamma_i \theta(f_i) + \sigma_i \quad \text{for all } i \in \mathcal{I} \setminus X.
\]

Again, if there is no cause for confusion, we will suppress \( \gamma \) and \( \sigma \) from the notation. Because of Corollary 3.3 we immediately obtain the following result, which addresses [Ko14, Rmk. 2.10].

**Proposition 3.12.** For \((X, \tau) \in \text{GSat}(A), \gamma \in \Gamma \) and \( \sigma \in \mathbb{C}^{I \setminus X} \), the defining relations of the universal enveloping algebra \( U(\mathfrak{t}) \) are given by (3.2-3.7).

We may view \( U(\mathfrak{t}) \) as a Hopf subalgebra of \( U(\mathfrak{g}) \) so that Lie algebra automorphisms of \( \mathfrak{g} \) lift to Hopf algebra automorphisms of \( U(\mathfrak{g}) \). Call two Hopf subalgebras \( B, B' \) of \( U(\mathfrak{g}) \) equivalent if there exists \( \phi \in \text{Aut}_{\text{Hopf}}(U(\mathfrak{g})) \) such that \( B' = \phi(B) \). Define
\[
\tilde{\Gamma} := \{ \gamma \in \Gamma | \gamma_i = 1 \text{ unless } i \in \mathcal{I}_\text{diff} \},
\]
\[
\Sigma := \{ \sigma \in \mathbb{C}^{I \setminus X} | \sigma_i = 0 \text{ unless } i \in \mathcal{I}_\text{nsf} \}.
\]

**Proposition 3.13.** Let \((X, \tau) \in \text{GSat}(A), \gamma \in \Gamma \) and \( \sigma \in \mathbb{C}^{I \setminus X} \). There exist \( \tilde{\gamma} \in \tilde{\Gamma} \) and \( \sigma' \in \Sigma \) such that \( U(\mathfrak{t}_\gamma)^\sigma \) is equivalent to \( U(\mathfrak{t}_{\tilde{\gamma}})^{\sigma'} \).

**Proof.** The existence of \( \tilde{\gamma} \) can be proven in an argument entirely analogous to the proof of [Ko14, Prop. 9.2 (i)]. It follows that \( U(\mathfrak{t}_\gamma)^\sigma \) is equivalent to \( U(\mathfrak{t}_{\tilde{\gamma}})^{\sigma} \) for some \( \tilde{\sigma} \in \mathbb{C}^{I \setminus X} \).

Regarding the existence of \( \sigma' \in \Sigma \), note that \( b_i; \tilde{\gamma} \in (\mathfrak{t}_i)_{\gamma} \) unless \( i \in \mathcal{I}_\text{nsf} \) owing to Prop. 3.4. Hence \( U(\mathfrak{t}_{\tilde{\gamma}})^{\tilde{\sigma}} \) is already generated by \( e_i, f_i \ (i \in X), h \in \mathfrak{h}^0, b_i; \tilde{\gamma}, 0 \) for \( i \in (\mathcal{I} \setminus X) \setminus \mathcal{I}_\text{nsf} \) and \( b_i; \tilde{\gamma}, 2 \) for \( i \in \mathcal{I}_\text{nsf} \). Hence we may take \( \sigma'_i = \tilde{\sigma}_i \) if \( i \in \mathcal{I}_\text{nsf} \) and \( \sigma'_i = 0 \) otherwise. \( \square \)

4. **Quantum pair algebras and the universal K-matrix revisited**

Assume the \( d_i \) are dyadic rationals and let \( \mathbb{K} \) be a quadratic closure of \( \mathbb{C}(q) \) where \( q \) is an indeterminate, so that \( q_i := q^{d_i} \in \mathbb{K} \) for all \( i \in \mathcal{I} \). The Drinfeld-Jimbo quantum group \( U_q(\mathfrak{g}) \) is an associative unital algebra over \( \mathbb{K} \) which quantizes the universal enveloping algebra \( U(\mathfrak{g}) \). It is generated by \( \{ E_i, F_i, t_i^{-1} \} \) where \( i \in \mathcal{I} \), satisfying the relations given in e.g. [Lu94, 3.1.1]. The Hopf algebra structure of the root space \( U_q(\mathfrak{g}) \), acting on the root space \( U_q(\mathfrak{g})_\alpha \) for \( \alpha \in \mathcal{Q} \) by multiplication by \( \chi(\alpha) \). Moreover, we have Lusztig’s automorphism \( T_i \) for \( i \in \mathcal{I} \), given as \( T_i^{(a)} \) in [Lu94, 37.1.3], which defines a braid group action on \( U_q(\mathfrak{g}) \). It satisfies \( T_i(U_q(\mathfrak{g})_\alpha) \subset U_q(\mathfrak{g})_{\alpha(i)} \) for all \( \alpha \in \mathcal{Q} \) and \( T_i(t_j) = t_j t_i^{-a_{ij}} \) for all \( j \in \mathcal{I} \). For \( X \subseteq \mathcal{I} \) with
Let \( w_X = s_{i_1} \cdots s_{i_k} \) a reduced decomposition we write \( T_X = T_{i_1} \cdots T_{i_k} \). A quantum analogue of the Chevalley involution is defined by

\[
\omega_q(E_i) = -t_i^{-1} F_i, \quad \omega_q(F_i) = -E_i t_i, \quad \omega_q(t_i^{\pm1}) = t_i^{\mp1}
\]

for \( i \in I \). Then \( \omega_q \) commutes with \( \text{Aut}(A) \) and with \( T_i \) for \( i \in I \), see [BK16, Lemma 7.1]. Using \( \tau(X) = X \) one straightforwardly checks that \( \tau \) commutes with \( T_X \).

We will now follow the approach of the papers [Ko14, BK15, BK16, Ko17, DK18] and highlight where a definition or formula needs to be changed in order to extend the theory to generalized Satake diagrams.

### 4.1. Quantum pair algebras

The quantum analog of the map \( \theta = \text{Ad}(w_X) \tau \omega \) is the map

\[
\theta_q = \theta_q(X, \tau) = T_X \tau \omega_q \in \text{Aut}_{\text{alg}}(U_q(g)).
\]

Note the absence of the factor \( \text{Ad}(s) \) from \( \theta_q \), cf. [Ko14, Def. 4.3] or [BK16, Def. 5.4 and Eq. (5.4)], which was present in \textit{ibid.} to guarantee that \( \theta_q \) specializes to the appropriate Lie algebra involution in the case \( (X, \tau) \in \text{Sat}(A) \), see [Ko14, Prop. 10.2]. In our notation, a suitable \( s \in H_q \) is given by \( \chi(\{1,1,\ldots,1\}) \), see (2.11).

The quantization of the fixed-point subalgebra in the formalism by [Ko14] relies on the presentation of \( g^0 \) in terms of generators given in [Ko14, Lemma 2.8]. Our \( (t(X, \tau)) \) with \( (X, \tau) \in \text{CDec}(A) \) by definition can be quantized to a right coideal subalgebra in the same way.

**Definition 4.1.** Let \( (X, \tau) \in \text{CDec}(A), \gamma \in (K^*)^{I/X} \) and \( \sigma \in K^{I^X} \). The quantum pair algebra \( B = B_{\gamma,\sigma}(X, \tau) \) is the coideal subalgebra generated by \( U_q(g_X), U_q(h)^{\theta_q} \) and the elements

\[
B_i = B_{i;\gamma,\sigma} = F_i + \gamma_i \theta_q(F_i t_i) t_i^{\gamma_i + 1} - \sigma_i t_i^{\gamma_i - 1} \quad \text{for all } i \in I \setminus X.
\]

**Remark 4.2.** Note that \( U_q(h)^{\theta_q} \) equals the algebra denoted \( U_q^{0'} \) in [Ko14]. The scalar \( \sigma_i \) is denoted \( s_i \) in [Le03, Eq. (2.4)] and [Ko14, Def. 5.1]; we use a different notation to avoid confusion with the simple reflections. The scalar \( \gamma_i \) corresponds to, but is not always equal to, the scalar \( d_i \) in [Le03, Eq. (2.4)] and the scalar \( c_i \) in [Ko14, Def. 5.1], to view the Lusztig-Kolb-Balagović formalism as a special case of our approach one should set, for all \( i \in I \setminus X \), \( \gamma_i = s(\alpha_{\tau(i)}) e_i \), see also [BK16, Eq. (7.7)].

Moreover, if \( (X, \tau) \in \text{GSat}(A) \) and the tuples \( \gamma, \sigma \) lie in the sets

\[
\Gamma_q = \{ \gamma \in (K^*)^{I/X} \mid \forall i \in I^* \gamma_i \neq \chi_{\tau(i)} \implies i \in \text{diff} \},
\]

\[
\Sigma_q = \{ \sigma \in K^{I^X} \mid \sigma_i = 0 \text{ unless } i \in I_{\text{nsf}} \},
\]

respectively, then according to [Ko14, Sec. 5.3 and Sec. 6] one obtains decompositions of \( B \) yielding the quantum analogue of (3.10), namely \( B \cap U_q(h) = U_q(h)^{\theta_q} \). The key condition for Satake diagrams, see (2.10), is only used in [Ko14, Proof of Lemma 5.11, Step 1], but it is clear that what is needed is precisely the weaker condition appearing in the definition of a generalized Satake diagram, see Definition 2.2. The rest of [Ko14] is applicable without change in the setting of generalized Satake diagrams; in particular in the specialization \( q \to 1 \) one recovers \( U(t) \), see [Ko14, Sec. 10].

### 4.2. Lusztig’s skew derivation

In [BK15] the bar involution for \( B \) is studied, following earlier work by [ES13] and [BW13] in the case of quantum symmetric pairs of \( gl_N \) type.

The only place in [BK15] which uses the defining condition of Satake diagrams or their classification is the proof of [BK15, Prop. 2.3], which is the statement that \( \sigma \tau \) fixes \( r_i(T_X(E_i)) \) for all \( i \in I \setminus X \). Here \( \sigma \) is the unique algebra anti-automorphism of \( U_q(g) \) which fixes \( E_i \) and \( F_i \) and inverts \( t_i \). Also, \( r_i \) is Lusztig’s \textit{(right) skew derivation}, see [Lu94, 1.2.13]; for \( i \in I \) it is the unique linear map \( r_i : U_q(n^+) \to U_q(n^+) \) such that

\[
r_i(xy) = q_i^{\mu(h_i)} r_i(x)y + xr_i(y)
\]

for all \( x, y \in U_q(n^+) \) with \( y \in U_q(g)_\mu \) (\( \mu \in Q^+ \)).

We denote \([x, y]_p := xy - pyx \) for \( x, y \in U_q(g) \) and \( p \in K \); note that \( \sigma([x, y]_p) = [\sigma(y), \sigma(x)]_p \). The definition of \( T_j \) implies that \( T_j(E_i) = E_i \) if \( a_{ji} = 0 \) and \( T_j(E_i) = [E_j, E_i]_{q^{-1}} \) if \( a_{ji} = -1 \).
In order to extend [BK15, Prop. 2.3] to generalized Satake diagrams (in fact, to compatible decorations), we provide a lemma that simplifies the proof drastically. Given \((X, \tau) \in \CDec(A)\), call a connected component of \(X\) simple if it is of the form \(\{j\}\) for some \(j \in I\) such that \(a_{ij} = a_{ji} = \{0, -1\}\) for all \(i \in I\).\(X\).

**Lemma 4.3.** Let \((X, \tau) \in \CDec(A)\) and \(i \in I \setminus X\). If \(\tilde{X}(i) = X_1 \cup \cdots \cup X_\ell\) is a decomposition into connected components then \(\ell \leq 1\) if \(i \neq \tau(i)\); if \(i = \tau(i)\) then all \(X_i\) are simple except at most one.

Suppose \(i = \tau(i)\) and \(\tilde{X}(i) \neq \emptyset\). If present, denote by \(Y\) the non-simple connected component of \(\tilde{X}(i)\); otherwise let \(Y\) be any simple connected component. If \(r_i(T_X(E_i))\) is fixed by \(\sigma\tau\) then so is \(r_i(T_X(E_i))\).

**Proof.** For the first part of the Lemma, note that the claim follows for generalized Satake diagrams from the classification of Satake diagrams, see e.g. [Ar62], and an inspection of Table 1. Since adding simple components does not change the statement, it is also true for compatible decorations.

The second part is proven by induction with respect to the number of \(i\)-simple components. If there are none, then \(\tilde{X}(i) = Y\) and the statement is true. Otherwise, by the induction hypothesis we may suppose \(\tilde{X}(i) = X' \cup \{j\}\) where \((\sigma\tau)(r_i(T_{X'}(E_i))) = r_i(T_{X'}(E_i))\), \(\{j\}\) is \(i\)-simple and \(a_{ij} = 0\) for all \(k \in X'\). Hence \(T_{\tilde{X}(i)} = T_{X'}T_j\) so that \(T_X(E_i) = T_{\tilde{X}(i)}(E_i) = T_{X'}(E_i)T_j\) if \(E_i\) is \(\tau(i)\)-simple. By (4.4) we have \(r_i(T_X(E_i)) = [E_j, r_i(T_{X'}(E_i))]_{q^{-2}}\). Since \(r_i(T_{X'}(E_i))\) lies in \(U_q(g_{X'})\) it commutes with \(E_j\). Hence

\[r_i(T_X(E_i)) = (1 - q^{-2})E_j r_i(T_{X'}(E_i)) = (1 - q^{-2})r_i(T_{X'}(E_i))E_j.\]

Note that \(\tau(j) = j\). By applying \(\sigma\tau\) we complete the proof. \(\square\)

**Proposition 4.4.** Let \((X, \tau) \in \CDec(A)\). Then for all \(i \in I \setminus X\), \(r_i(T_X(E_i))\) is fixed by \(\sigma\tau\).

**Proof.** The proof is essentially casework, but first we make some observations.

(i) Since \(T_X(E_i) = T_{\tilde{X}(i)}(E_i)\) we may assume that \(\{i, \tau(i)\}\) is the only \(\tau\)-orbit outside \(X\).

(ii) We may assume \(X\) is nonempty as otherwise \(r_i(T_X(E_i)) = 1\).

(iii) By Lemma 4.3 it suffices to prove the statement in the case that \(X\) is connected.

(iv) If \(|X| = 1\), we write \(X = \{j\}\) with \(\tau(j) = j\). Then \(T_X(E_i) = T_j(E_i) \in U_q(g)_{\sigma_{\tau}(\alpha_3) \cap U_q(n^+)}\). Hence \(r_i(T_X(E_i)) \in U_q(g)_{\sigma_{\tau}(\alpha_3) \cap U_q(n^+)}\) becomes empty, so it is fixed by \(\sigma\tau\).

(v) In [BK15, Proof of Prop. 2.3] the statement was proved for all Satake diagrams.

Hence it suffices to prove the statement for those diagrams in Table 1 where the node \(i\) is the only node outside \(X\), \(X\) is connected and \(|X| > 1\). There is one infinite family of diagrams satisfying this condition as well as some exceptional diagrams. The infinite family is given by the diagrams \(\begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \end{array}\) with \(n \geq 3\). In this case the proof is identical to the proof for the type BII case in [BK15, Prop. 2.3] (it does not use the values \(a_{n-1n}\) and \(a_{n,n-1}\)).

The remaining exceptional cases are

![Graphical representation of exceptional cases](image)

We give here the proof for the last case which is very much in the spirit of the proofs in [BK15, Proof of Prop. 2.3]; the proofs for the other three cases are similar and are left to the reader.

We label the nodes as \(\begin{array}{c} 1 \bullet 2 \bullet 3 \bullet 4 \end{array}\) and assume \(d_1 = d_2 = 2\) and \(d_3 = d_4 = 1\) for convenience. The reduced decompositions \(w_X = (s_2s_3s_2s_3s_2)(s_4s_3s_4) = (s_4s_3s_4)(s_2s_3s_2s_2s_3s_2)\) yield

\[(4.5) \quad T_X(E_1) = (T_2T_3T_2T_4T_3T_2)(E_1) = (T_2T_3T_2T_4T_3)(E_1).\]

From the first expression we readily obtain

\[T_X(E_1) = [(T_2T_3T_2)(E_2), [(T_3T_2)(E_2), [E_2, E_1]_{q^{-2}}, q^{-2}q^{-2}].\]

Now note that \((s_4s_3s_2s_3s_2)(\alpha_2) = \alpha_2\) and \(s_3s_4s_3s_2s_4s_3\) and \(s_3s_2s_3\) are reduced elements in \(W\). Appealing to [Jan96, Prop. 8.20] we arrive that

\[T_X(E_1) = [(T_4^{-1}T_3^{-1})(E_2), [T_3^{-1}(E_2), [E_2, E_1]_{q^{-2}}, q^{-2}]_{q^{-2}}.\]
so that (4.4) implies
\[ r_1(T_X(E_1)) = (1 - q^{-4})[(T_4^{-1}T_3^{-1})(E_2), [T_3^{-1}(E_2), E_2]_{q^{-4}}]. \]

Applying \( \sigma \tau \) and using \( T_i \sigma = \sigma T_i^{-1} \) (see e.g. [Lu94, 37.2.4]) we obtain
\[ (\sigma \tau)(r_1(T_X(E_1))) = (1 - q^{-4})[[E_2, T_3(E_2)], (T_4T_3(E_2))]_{q^{-4}}. \]

Consider the nested commutator \([T_3(E_2), [T_4T_3(E_2), E_2]].\) A direct computation gives \([T_3(E_2), E_2] = (q^2 - 1)[2]q^{-1}T_2^{-1}(E_3)^2.\) This yields
\[ [T_3(E_2), [T_4T_3(E_2), E_2]] = \frac{q^2 - 1}{[2]q}T_2^{-1}([T_2T_3(E_2), T_4(E_3)^2]). \]

The reduced elements \( s_1s_2s_3 \) and \( s_3s_4 \) map \( \alpha_2 \) to itself and \( \alpha_3 \) to \( \alpha_4 \), respectively, so that \( (T_2T_3)(E_2) = T_3^{-1}(E_3) \) and \( (T_4T_3)(E_3) = E_4 \) by [Jan96, Prop. 8.20]. Combining this with \([E_2, E_4] = 0\) we obtain
\[ [T_3(E_2), [T_4T_3(E_2), E_2]] = \frac{q^2 - 1}{[2]q}T_2^{-1}([T_3^{-1}(E_2), T_4(E_3)^2]) = 0. \]

Using this, (4.6) yields
\[ (\sigma \tau)(r_1(T_X(E_1))) = (1 - q^{-4})[[E_2, T_3(E_2), (T_4T_3)(E_2)]_{q^{-4}}. \]
\[ = (1 - q^{-4})(T_3T_4T_3) \left( ([T_4^{-1}T_3^{-1}(E_2), [T_3^{-1}(E_2), E_2]_{q^{-4}}) \right) = (T_3T_4T_3)(r_1(T_X(E_1))) \]
where we have used \( T_3T_4T_3 = T_4T_3T_4. \) Now \( r_i \) and \( T_i \) commute if \( a_{ij} = 0 \) so that \( (\sigma \tau)(r_1(T_X(E_1))) = r_1((T_4T_3T_4T_X)(E_1)) \) and by virtue of (4.5) the proof is complete.

**4.3. The universal K-matrix.** In [BK16] the universal K-matrix for \( B \) is constructed for \((X, \tau) \in \text{Sat}(A); \) in particular the relations (1.1) and (1.3) are derived. In [BK16, Proof of Lemma 6.4] the defining condition of Satake diagrams is used, but only the defining condition of generalized Satake diagrams is needed. For completeness, we restate some key conditions in terms of \( \gamma \). Condition [BK16, Eq. (5.17)] is equivalent to
\[ \gamma_{\tau(i)} = \zeta(\alpha_i)q_i^{\delta(\alpha_i) - 2\rho_X}(h_i)\gamma_{\tilde{\tau}(i)}, \]
where \( \rho_X \) is the Weyl vector of \( q_X \) and \( \bar{\ } \) denotes the bar involution of \( U_q(g) \), the algebra automorphism fixing \( E_i, F_i \) and inverting \( t_i^{-1} \) and \( q \). Then [BK16, Eq. (7.14)] and hence [BK16, Eq. (9.8)] are equivalent to
\[ T_{w_X}(E_{\gamma(i)}) = \zeta(\alpha_i)q_i^{2\rho_X(h_i)}T_{w_X}^{-1}(E_{\tau(i)}), \]
\[ \gamma_{\tau(i)}T_{w_X}(E_{\gamma(i)}) = q_i^{-\delta(\alpha_i)(h_i)}\gamma_{\tau(i)}T_{w_X}^{-1}(E_{\tau(i)}), \]
respectively, so that the scalar \( \rho_i \) appearing in [BK16, Lemma 9.3] equals \( q_i^{-\delta(\alpha_i)(h_i)}\gamma_{\tau(i)}. \)

In [Ko17] it is shown that \( \mathcal{K} \) satisfies (1.2) and the centre of \( B \) is described in terms of \( \mathcal{K} \) without using the defining condition of Satake diagrams or a case-by-case analysis; it follows the results remain valid for \((X, \tau) \in \text{GSat}(A). \)

This is also essentially the case for the paper [DK18] which establishes an elegant factorization property of the quasi K-matrix in terms of the restricted Weyl group of \( g \). Sections 2.2 and 2.3 in [ibid.] entail an analysis of the restricted Weyl group and restricted root system following [Lu76]. In reference to a comment in [DK18, between Eqs. (2.9) and (2.10)], note that also for all \((X, \tau) \in \text{GSat}(A) \setminus \text{Sat}(A)\) the set \( X \) is invariant under \( \tau_0; \) this follows from Table 1. The upshot of this in [DK18] is that \( \tau_0 \) stabilizes \( X \) for all \( i \in I^* \). This is used to derive that the \( \tilde{s}_i = w_X w_X[i] \) form a Coxeter system for the group they generate. Alternatively, this result follows from Theorem 2.4 (vi) for all generalized Satake diagrams.

**A. Deriving Serre relations for \( \mathfrak{k} \)**

The following three technical lemmas are used to derive the key equation (3.7). It is convenient to introduce the notation \( Q_X = \sum_{i \in X} \mathbb{Z} \alpha_i \) and \( Q_X^+ := Q^+ \cap Q_X. \)
Lemma A.1. Let \((X, \tau) \in \mathrm{CDec}(A)\) and \(\gamma \in (\mathbb{C}^\times)^{I \setminus X}\). For all \(i \in X, j \in I\) and \(m \in \mathbb{Z}_{\geq 1}\),
\[
\mathrm{ad}(b_i)^m(b_j) = \begin{cases} 
\mathrm{ad}(f_i)^m(f_j) + \gamma_j \theta(\mathrm{ad}(f_i)^m(f_j)) & \text{if } j \notin I \setminus X, \\
\mathrm{ad}(f_i)^m(f_j) & \text{if } j \in X.
\end{cases}
\]

Proof. This follows immediately from (2.7) and the fact that \(\theta\) is a Lie algebra automorphism. \(\square\)

Lemma A.2. Let \((X, \tau) \in \mathrm{CDec}(A)\) and \(\gamma \in (\mathbb{C}^\times)^{I \setminus X}\). For all \(i \in I \setminus X, j \in X\) and \(m \in \mathbb{Z}_{\geq 1}\),
\[
\mathrm{ad}(b_i)^m(b_j) = \mathrm{ad}(f_i)^m(f_j) + \gamma_i \theta(\mathrm{ad}(f_i)^m(f_j)) + \LO_{ij}(m)
\]
where
\[
\LO_{ij}(m) = \begin{cases}
(1 + \zeta(\alpha_i)) \gamma_i [\theta(f_i), [f_i, f_j]] \in \n_X & \text{if } \tau(i) = i, w_X(\alpha_i) - \alpha_i - \alpha_j \in \Phi^+, m = 2, \\
-\gamma_i (2 h_i - a_i h_j) & \text{if } \tau(i) = i, w_X(\alpha_i) = \alpha_i + \alpha_j, m = 2, \\
-3(2 + a_{ij}) \gamma_i (f_i - \theta(f_i)) & \text{if } \tau(i) = i, w_X(\alpha_i) = \alpha_i + \alpha_j, m = 3, \\
-6a_{ij}(2 + a_{ij}) \gamma_i^2 e_j & \text{if } \tau(i) = i, w_X(\alpha_i) = \alpha_i + \alpha_j, m = 4, \\
0 & \text{otherwise}.
\end{cases}
\]

Proof. By induction with respect to \(m\). For \(m = 1\), (2.7) implies
\[
\mathrm{ad}(b_i)^1(b_j) = [f_i + \gamma_i \theta(f_i), f_j] = \mathrm{ad}(f_i)^1(f_j) + \gamma_i \theta(\mathrm{ad}(f_i)(f_j)) + \LO_{ij}(1)
\]
with \(\LO_{ij}(1) = 0\) as required. Now assume \(m \in \mathbb{Z}_{> 1}\) and suppose the statement holds for all smaller values. Then, by virtue of the induction hypothesis, the fact that \(\theta\) is a Lie algebra automorphism and (2.7), we find
\[
\mathrm{ad}(b_i)^m(b_j) = [b_i, \mathrm{ad}(b_i)^{m-1}(b_j)]
\]
\[
= [f_i + \gamma_i \theta(f_i), \mathrm{ad}(f_i)^{m-1}(f_j) + \gamma_i^{m-1} \theta(\mathrm{ad}(f_i)^{m-1}(f_j)) + \LO_{ij}(m-1)]
\]
\[
= \mathrm{ad}(f_i)^m(f_j) + \gamma_i \theta(\mathrm{ad}(f_i)^m(f_j))
\]
\[
+ \gamma_i \theta(f_i, \mathrm{ad}(f_i)^{m-1}(f_j)) + \gamma_i^{m-1} [f_i, \theta(\mathrm{ad}(f_i)^{m-1}(f_j))] + [b_i, \LO_{ij}(m-1)].
\]

Using (2.8) we have \(\theta^2(f_i) = \zeta(\alpha_i) f_i\) so that
\[
\LO_{ij}(m) = \gamma_i [\theta(f_i), \mathrm{ad}(f_i)^{m-1}(f_j)] + \zeta(\alpha_i) \gamma_i^{m-1} \theta([\theta(f_i), \mathrm{ad}(f_i)^{m-1}(f_j)]) + [b_i, \LO_{ij}(m-1)].
\]

Suppose that \([\theta(f_i), \mathrm{ad}(f_i)^{m-1}(f_j)] \neq 0\). Then \(w_X(\alpha_{\tau(i)}) - (m - 1) \alpha_i - \alpha_j \in \Phi^+ \cup \{0\}\). Now \(\Phi = \Phi^+ \cup \Phi^-\) implies that \(\tau(i) = i \in \hat{X}(i)\).

If \(w_X(\alpha_{\tau(i)}) - (m - 1) \alpha_i - \alpha_j \in \Phi^+\) we have \(\tau(i) = i, m = 2\); since \(w_X(\alpha_i) - \alpha_i - \alpha_j \in Q_X^+\) it follows that \([\theta(f_i), [f_i, f_j]] \in n_X\). The claimed expression for \(\LO_{ij}(2)\) follows immediately from (A.1); those for \(\LO_{ij}(m)\) with \(m > 2\) from (3.2).

If \(w_X(\alpha_{\tau(i)}) - (m - 1) \alpha_i - \alpha_j \in \Phi^- \cup \{0\}\), \(\tau(i) = i\) and \(w_X(\alpha_i) \leq (m - 1) \alpha_i + \alpha_j\). Hence \(\hat{X}(i) = \{j\}\) and \(a_{ji} < 0\). In this case we readily obtain
\[
w_X(\alpha_{\tau(i)}) - (m - 1) \alpha_i - \alpha_j = (2 - m) \alpha_i - (1 + a_{ji}) \alpha_j.
\]

From \(\Phi = \Phi^+ \cup \Phi^-\) it follows that \(a_{ji} = -1\). Now \(\mathbb{Z}_{\alpha_i} \cap \Phi = \{\pm \alpha_i\}\) implies that \(m \in \{2, 3\}\). We straightforwardly compute
\[
[\theta(f_i), \mathrm{ad}(f_i)^{m-1}(f_j)] = \begin{cases}
(a_{ij} h_j - h_i) & \text{if } m = 2, \\
-2(1 + a_{ij}) f_i & \text{if } m = 3,
\end{cases}
\]
and the claimed expressions for \(\LO_{ij}(m)\) readily follow. \(\square\)

For \(i, j \in I\) and \(r, s \in \mathbb{Z}\) such that \(0 \leq r \leq [m/2]\) define \(p_{ij}^{(r, m)} \in \mathbb{Z}\) recursively by
\[
(A.2) \quad p_{ij}^{(0, m)} = -1, \quad p_{ij}^{(m+1, m)} = 0, \quad p_{ij}^{(r+2, m+2)} = p_{ij}^{(r+1, m+1)} - (m + 1)(m + a_{ij}) p_{ij}^{(r-1, m)}.
\]
Lemma A.3. Let \((X, \tau) \in \mathbb{CDec}(A)\) and \(\gamma \in (\mathbb{C}^\times \setminus X) \cap X\). For all \(i, j \in \Gamma \setminus X\) such that \(i \neq j\) and \(m \in \mathbb{Z}_{\geq 0}\),
\[
\text{ad}(b_i)^m(b_j) = \text{ad}(f_i)^m(f_j) + \theta(\text{ad}(f_i)^m(f_j)) + \text{LO}_{ij}(m)
\]
where
\[
\text{LO}_{ij}(m) = \begin{cases} 
(\gamma_i + \zeta(\alpha_i)\gamma_j)[\theta(f_i), f_j] \in \mathfrak{n}_X^\mathbb{C} & \text{if } \tau(i) = j, w_X(\alpha_i) - \alpha_i \in \Phi^+, m = 1, \\
\gamma_i h_i - \gamma_j h_j & \text{if } \tau(i) = j, w_X(\alpha_i) = \alpha_i, m = 1, \\
2 (\gamma_j - a_{ij} \gamma_i) f_i - \gamma_i (\gamma_i - a_{ij} \gamma_j) e_j & \text{if } \tau(i) = j, w_X(\alpha_i) = \alpha_i, m = 2, \\
\sum_{r=1}^{\lfloor m/2 \rfloor} P_{ij}^{(r, m)} \gamma_i^r \text{ad}(b_i)^{m-2r}(b_j) & \text{if } \tau(i) = i, w_X(\alpha_i) = \alpha_i, \\
0 & \text{otherwise.}
\end{cases}
\]

Proof. As before we apply induction with respect to \(m\). For \(m = 0\) we have
\[
\text{ad}(b_i)^0(b_j) = b_j = f_j + \gamma_j \theta(f_j) = \text{ad}(f_i)^0(f_j) + \gamma_i^0 \gamma_j \theta(\text{ad}(f_i)^0(f_j)) + \text{LO}_{ij}(0)
\]
with \(\text{LO}_{ij}(0) = 0\) as required. Now assume \(m \in \mathbb{Z}_{\geq 0}\) and suppose the statement holds for all smaller values. Then, by the induction hypothesis,
\[
\text{ad}(b_i)^m(b_j) = [b_i, \text{ad}(b_i)^{m-1}(b_j)]
\]
\[
= [f_i + \gamma_i \theta(f_i), \text{ad}(f_i)^{m-1}(f_j)] + \gamma_i^{m-1} \gamma_j \theta(\text{ad}(f_i)^{m-1}(f_j)) + \text{LO}_{ij}(m - 1).
\]
Rearranging terms and using that \(\theta\) is a Lie algebra automorphism we obtain
\[
\text{ad}(b_i)^m(b_j) = \text{ad}(f_i)^m(f_j) + \gamma_i^{m-1} \gamma_j \theta(\text{ad}(f_i)^m(f_j)) + \text{LO}_{ij}(m)
\]
where
\[
\text{LO}_{ij}(m) = \gamma_i \left[\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)\right] + \gamma_i^{m-1} \gamma_j \left[f_i, \theta(\text{ad}(f_i)^{m-1}(f_j))\right] + [b_i, \text{LO}_{ij}(m - 1)].
\]
Using (2.8) we obtain
\[
\text{LO}_{ij}(m) = \gamma_i \left[\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)\right] + \zeta(\alpha_i) \gamma_i^{m-1} \gamma_j \theta\left(\left[\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)\right]\right) + [b_i, \text{LO}_{ij}(m - 1)].
\]
If \(\left[\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)\right] \neq 0\) then \(w_X(\alpha_{\tau(i)}) - (m - 1) \alpha_i - \alpha_j \in \Phi \cup \{0\}\).

If \(w_X(\alpha_{\tau(i)}) - (m - 1) \alpha_i - \alpha_j \in \Phi^+\) we must have \(j = \tau(i), X(i) \neq \emptyset, m = 1\); since \(w_X(\alpha_{\tau(i)}) - \alpha_j \in Q_X\) it follows that \(\left[\theta(f_i), f_j\right] \in \mathfrak{n}_X^\mathbb{C}\). The expression for \(\text{LO}_{ij}(1)\) follows from (A.3); \(\text{LO}_{ij}(m) = 0\) with \(m > 1\) is a consequence of (3.2).

Now suppose \(w_X(\alpha_{\tau(i)}) - (m - 1) \alpha_i - \alpha_j \in \Phi^- \cup \{0\}\). It follows that \(X(i) = \emptyset\), so \(\zeta(\alpha_i) = 1\), and \(\tau(i) \in \{i, j\}\). If \(\tau(i) = j\) then \(\mathbb{Z} \alpha_i \cap \Phi = \{\pm \alpha_i\}\) implies that \(m \in \{1, 2\}\). Furthermore, \(\theta(f_i) = -e_j\) and \(a_{ij} = a_{ji}\). Now (A.3) implies, as required, \(\text{LO}_{ij}(1) = \gamma_j h_i - \gamma_i h_j\),
\[
\text{LO}_{ij}(2) = \gamma_i \gamma_j \theta([-e_j, [f_i, f_j]]) + \gamma_i [-e_j, [f_i, f_j]] + [b_i, \text{LO}_{ij}(1)]
\]
\[
= \gamma_i \gamma_j \theta([h_j, f_i]) + \gamma_i [h_j, f_i] + [\gamma_i h_j - \gamma_j h_i, f_i - \gamma_i e_j]
\]
\[
= 2 ((\gamma_j - a_{ij} \gamma_i) f_i - \gamma_i (\gamma_i - a_{ij} \gamma_j) e_j)
\]
and \(\text{LO}_{ij}(m) = 0\) if \(m > 2\).

It remains to deal with the case \(X(i) = \emptyset\) and \(\tau(i) = i\), in which case \(\theta(f_i) = -e_i\). A straightforward computation gives
\[
\left[\theta(f_i), \text{ad}(f_i)^{m-1}(f_j)\right] = (m - 1)(m - 2 + a_{ij}) \text{ad}(f_i)^{m-2}(f_j).
\]
By virtue of the induction hypothesis, (A.3) simplifies to
\[
\text{LO}_{ij}(m) = (m - 1)(m - 2 + a_{ij}) \gamma_i (\text{ad}(b_i)^{m-2}(b_j) - \text{LO}_{ij}(m - 2)) + [b_i, \text{LO}_{ij}(m - 1)],
\]
from which the recursion (A.2) follows straightforwardly. \(\square\)
References


