Mortality and Edge-to-Edge Reachability are Decidable on Surfaces

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ABSTRACT

The mortality problem for a given dynamical system S consists of determining whether every trajectory of S eventually halts. In this work, we show that this problem is decidable for the class of piecewise constant derivative systems on two-dimensional manifolds, also called surfaces (PCD_{2m}). Two closely related open problems are point-to-point and edge-to-edge reachability for PCD_{2m}.

Building on our technique to establish decidability of mortality for PCD_{2m}, we show that the edge-to-edge reachability problem for these systems is also decidable. In this way we solve the edge-to-edge reachability case of an open problem due to Asarin, Mysore, Pnueli and Schneider [4]. This implies that the interval-to-interval version of the classical open problem of reachability for regular piecewise affine maps (PAMs) is also decidable. In other words, point-to-point reachability for regular PAMs can be effectively approximated with arbitrarily precision.

CCS CONCEPTS

• **Theory of computation → Timed and hybrid models**.

KEYWORDS

mortality, reachability, decidability, dynamical systems

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1 INTRODUCTION

A trajectory of a dynamical system is said to be *mortal* if it halts in a finite number of steps. On the other hand, such a trajectory is said to be *immortal* if it never stops. One of the most fundamental computational problems in dynamical system theory is the *mortality problem* [7], which in general terms, can be stated as follows: Given the specification of a dynamical system S, is it the case that all trajectories in S are mortal?

The mortality problem is relevant to the field of program termination, and it has been studied in different contexts and in different variants [9, 10, 22]. It may also be regarded as a generalisation of the halting problem to the realm of dynamical systems. The noted result from Blondel et al. states that mortality is undecidable for piecewise affine functions on a plane [10], and identifying subclasses of dynamical systems for which it becomes decidable has attracted the attention of the community [8, 10].

Closely related to mortality is the *reachability problem*: does exist a trajectory starting at a given initial state which evolves to reach a given final state [1]. The reachability problem is undecidable in general, and it is only decidable for a limited number of classes of dynamical systems [11].

In their seminal paper, Asarin et al. aimed at understanding what is the simplest class of hybrid systems for which reachability is undecidable. They considered hybrid systems with one, two and three dimensions that span the boundary between decidability and undecidability for the reachability problem [4]. In particular, they studied piecewise constant derivative systems (PCDs) on two-dimensional *manifolds* and showed that these systems are equivalent with respect to reachability to regular one-dimensional *piecewise affine maps*. That is, it is an important open problem to determine whether reachability is decidable for both systems [4, 13, 14].

In this work we consider piecewise constant derivative systems on two-dimensional manifolds, also called surfaces, and we use the abbreviation PCD_{2m} to denote such systems. The first result of this work (Theorem 3.1) is decidability of the mortality problem for PCD_{2m} systems.

In the context of PCD_{2m}, reachability comes into two flavours: point-to-point reachability and edge-to-edge reachability. Building on the new technique we developed to establish decidability of mortality for PCD_{2m}, we show that the edge-to-edge reachability problem for these systems is also decidable (Theorem 4.2). In this way we solve the edge-to-edge reachability case of an open problem due to Asarin, Mysore, Pnueli and Schneider [4]. Previously, this problem was only known to be decidable under certain restrictions [20].

Decidability of reachability for one-dimensional PAMs is a long-standing open problem, and that is why these systems are often used as the reference model to demonstrate the openness of the reachability query [4, 12, 14]. We extend our results further and show that the interval-to-interval version of the reachability problem is decidable for regular PAMs (Theorem 5.4). This way the point-to-point reachability problem for these systems can be effectively approximated with arbitrarily precision.

The approaches for proving decidability of reachability often rely on the topological properties of the systems under consideration.

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Thus, the proof of decidability of reachability for PCDs on a plane proposed in [15] is based on the Jordan curve theorem for $\mathbb{R}^2$ which states that every non-self-crossing closed curve divides the plane into an 'interior' region bounded by the curve and an 'exterior' region containing all other points of $\mathbb{R}^2$, so that every continuous path connecting a point of one region to a point of the other intersects with that curve somewhere. As a result, each trajectory in such a system eventually forms a simple cycle of border edges. As Jordan's theorem is not applicable to manifolds, we use topological properties of trajectories on surfaces that allow decomposing them into maximal connected components consisting of trajectories with similar behaviour [3, 16].

While the aim of the paper is to contribute to better understanding the boundary between decidable and undecidable systems, it is worth mentioning that systems on manifolds have important practical applications. Areas where surfaces arise include among others biological systems (modelling processes on cell membranes [19]), robotics (the configuration space of a robotic arm) and learning algorithms (finding a low-dimensional parameterization of high-dimensional data [24]).

The rest of the paper is organised as follows. In Section 2 we introduce preliminaries. In Section 3 and Section 4 we prove decidability of mortality and edge-to-edge reachability for PCD_{2m} systems. In Section 5 we show that the interval-to-interval reachability is decidable for regular one-dimensional PAMs. Section 6 contains concluding remarks.

2 PRELIMINARIES

In this section, we define the notion of two-dimensional piecewise constant derivative systems following closely the notation used in [15]. Later on, we will generalise this notion to the context of closed surfaces and then prove our main results.

2.1 Definitions

We write $\mathbb{R}^2$ to denote the *2-dimensional Euclidean space*. *Points* in $\mathbb{R}^2$ are denoted by bold letters such as $\vec{x}$ or $\vec{y}$. An *open half-space* in $\mathbb{R}^2$ is the set of points $\vec{x} \in \mathbb{R}^2$ satisfying some linear inequality of the form $\vec{a} \cdot \vec{x} + \vec{b} < 0$, for some rational vectors $\vec{a}$ and $\vec{b}$. A *convex open polygonal set* p is the intersection of a finite number of open half-spaces.

Given a set $\mathsf{X} \subseteq \mathbb{R}^2$, we let $\mathsf{cl}(\mathsf{X})$ denote the *topological closure* of X, and let $\mathsf{int}(\mathsf{X})$ denote the *interior*[1] of X. That is to say, the set of points $\vec{x} \in \mathsf{X}$ such that for some $\varepsilon > 0$, the ε-neighbourhood $N_\varepsilon(\vec{x})$ of $\vec{x}$ is contained in X. The *boundary* of a convex open polygonal set p is defined as $\mathsf{bd}(p) = \mathsf{cl}(p) \backslash p$.

A *finite polygonal partition* of a set $\mathsf{X} \subseteq \mathbb{R}^2$ is a set $\mathsf{P} = \{p_1, \ldots, p_k\}$ of convex open polygonal sets, called *regions*, such that: (1) $p_i \neq \emptyset$ for each $1 \leq i \leq k$; (2) $p_i \cap p_j = \emptyset$ for each $1 \leq i, j \leq k$ such that $i \neq j$; (3) $\bigcup_{i=1}^{k} \mathsf{cl}(p_i) = \mathsf{X}$.

An edge of P is a set of the form $e = \mathsf{int}(\mathsf{cl}(p_i) \cap \mathsf{cl}(p_j))$ for some $p_i, p_j \in \mathsf{P}$ with $i \neq j$, and $\mathsf{int}(\mathsf{cl}(p_i) \cap \mathsf{cl}(p_j)) \neq \emptyset$.

For a line segment e on the plane we denote by $\mathsf{int}(e)$ its relative interior, that is e without its endpoints.

[1] Note that for a line segment e on the plane we use $\mathsf{int}(e)$ to denote its relative interior, that is e without its endpoints.

We let $\mathsf{E}(\mathsf{P})$ denote the set of *edges* of P. A *vertex* of P is a singleton of the form $v = \mathsf{cl}(e_i) \cap \mathsf{cl}(e_j)$, where $e_i, e_j \in \mathsf{E}(\mathsf{P})$ with $i \neq j$, and $\mathsf{cl}(e_i) \cap \mathsf{cl}(e_j) \neq \emptyset$.

We let $\mathsf{V}(\mathsf{P})$ denote the set of *vertices* of P. We say that $\mathsf{Bd}(\mathsf{P}) = \mathsf{E}(\mathsf{P}) \cup \mathsf{V}(\mathsf{P})$ is the set of *border elements* of P. We note that the set $\mathsf{P} \cup \mathsf{Bd}(\mathsf{P})$ forms a partition of X.

We say that P is a *triangulation* if $|\mathsf{V}(p) \cap \mathsf{cl}(p)| = 3$ for every $p \in \mathsf{P}$.

We define the *border elements* of a region $p \in \mathsf{P}$ as

$$\mathsf{Bd}(p) = \{b \in \mathsf{Bd}(\mathsf{P}) \mid b \subseteq \mathsf{cl}(p)\}.$$

Finally, we let $\mathsf{bd}(\mathsf{P}) = \bigcup_{p \in \mathsf{P}} \mathsf{bd}(p)$ be the set of *border points* of P, where $\mathsf{bd}(p)$ is the boundary of p defined above.

2.2 Piecewise Constant Derivative Systems on a Plane

A 2*-dimensional piecewise constant derivative system* (2-PCD) is defined as a finite set of regions, together with a map that associates a constant vector field to each region, and a map that specifies which boundary elements belong to each region.

We note that each boundary element is assigned to a unique region. Since the vector field associated to each region is constant, this field can be specified by a single vector in R^2. Intuitively, the vector associated to each region specifies the direction a particle would follow when entering the region from one of its borders.

Definition 2.1 (2-PCD). A two-dimensional piecewise constant derivative system (2-PCD) is a triple $\mathsf{H} = (\mathsf{P}, \varphi, \psi)$ where P is a finite polygonal partition of $\mathbb{R}^2$, $\varphi : \mathsf{P} \to \mathbb{R}^2$ is a function that assigns a vector $\varphi(p)$ in $\mathbb{R}^2$ to each region $p \in \mathsf{P}$, and $\psi : \mathsf{Bd}(\mathsf{P}) \to \mathsf{P}$ is a function such that $e \subseteq \mathsf{cl}(\psi(e))$ for every $e \in \mathsf{Bd}(\mathsf{P})$.

Let $\mathsf{H} = (\mathsf{P}, \varphi, \psi)$ be a 2-PCD, and $p \in \mathsf{P}$ be a region and $e \in \mathsf{Bd}(p)$ be an edge of p. We say that e is an *input edge* for p if for any $x \in e$ there is some $t > 0$ such that $x + t \cdot \varphi(p) \in p$. We say that e is an *output edge* for p if for each $x \in e$, there exists some $t < 0$ such that $x + t \cdot \varphi(p) \in p$.

Let $\vec{x} = \mathsf{cl}(e) \cap \mathsf{cl}(e')$ for some edges $e, e' \in \mathsf{Bd}(p)$. We say that $\vec{x}$ is an *input vertex* for p if both e, e' are input edges for p; we say that $\vec{x}$ is an *output vertex* for p if both e, e' are output edges for p. Finally, we say that $\vec{x}$ is *neutral*, if it is neither an input nor an output edge for p.

We denote by $\mathsf{In}(p) \subseteq \mathsf{Bd}(p)$ and $\mathsf{Out}(p) \subseteq \mathsf{Bd}(p)$ the sets of input and output border elements (edges and vertices) of p respectively.

Definition 2.2 (Step). Let $\mathsf{H} = (\mathsf{P}, \varphi, \psi)$ be a 2-PCD, and $\vec{x}$ and $\vec{x'}$ be two distinct points in $\mathbb{R}^2$. We say that the pair $(\vec{x}, \vec{x'})$ is a *step* if there is a region $p \in \mathsf{P}$ and a $t > 0$ such that the following holds: (1) $\vec{x}' = \vec{x} + t \cdot \varphi(p)$; (2) $\vec{x} \in e, \vec{x'} \in e'$ such that $\psi(e) = \psi(e') = p$; (3) $\vec{x}'' = \vec{x} + t' \cdot \varphi(p) \in \mathsf{int}(p)$ for each $0 < t' < t$.

Intuitively, $(\vec{x}, \vec{x'})$ is a step if $\vec{x}$ and $\vec{x'}$ are points in the border of some region p and the line by $\vec{x}$ and $\vec{x'}$ is fully contained in p.

Definition 2.3 (Trajectory). Let $\vec{x}_0$ be a point in $\mathsf{bd}(\mathsf{P})$, and let $\ell \in \mathbb{N}$. A *trajectory* of length ℓ rooted at $\vec{x}_0$ is a sequence $\tau_{\vec{x}_0}^{\ell} = \vec{x}_0 \vec{x}_1 \ldots \vec{x}_\ell$ where for each $i \in \{1, \ldots, \ell\}$, $(\vec{x}_i, \vec{x}_{i+1})$ is a step in H.

Example 2.4. Now we consider a simple 2-PCD $\mathsf{H} = (\mathsf{P}, \varphi, \psi)$ depicted in Figure 1. The polygonal partition P consists of eight regions, and each region is assigned some dynamics as it is shown in Figure 1(a). The set $\mathsf{V}(\mathsf{P})$ consists of seventeen vertices labelled from a to s. The set $\mathsf{E}(\mathsf{P})$ includes twenty four edges. And finally, the function ψ assigns an element of the set $\mathsf{Bd}(\mathsf{P}) = \mathsf{V}(\mathsf{P}) \cup \mathsf{E}(\mathsf{P})$ to a region. An example of a trajectory of length 4 is show in Figure 1(b).

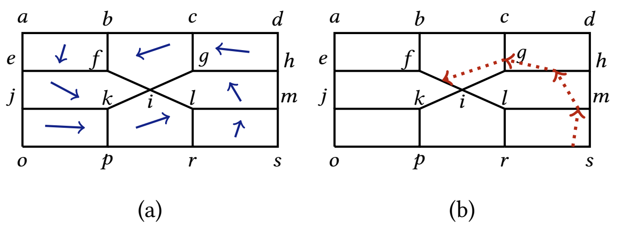


Figure 1: (a) An example of a 2-PCD with eight regions. (b) An example of a trajectory of size four. Each arrow corresponds to a step.

Note that for each point $\vec{x}_0$, and each $\ell \in \mathbb{N}$, such a trajectory is unique. Furthermore, we say that a point $\vec{x}_f \in \mathsf{bd}(\mathsf{P})$ is *reachable* from $\vec{x}_0$ if $\vec{x}_f$ belongs to the trajectory $\tau_{\vec{x}_0}$.

Next, we define the notion of *signature* of a trajectory. Similar definitions have been considered in [6].

Definition 2.5 (Signature). Let $\mathsf{H} = (\mathsf{P}, \varphi, \psi)$ be a 2-PCD system, $\vec{x}_0$ be a point in $\mathsf{bd}(\mathsf{P})$, and $\tau^{\ell}_{\vec{x}_0} = \vec{x}_0\vec{x}_1 \dots \vec{x}_\ell$ be the trajectory of size ℓ rooted at $\vec{x}_0$. The signature of $\tau^{\ell}_{\vec{x}_0}$ is the sequence $\sigma^{\ell}_{\vec{x}_0} = b_0, b_1, \dots, b_\ell$ where for each $i \in \{0, 1, \dots, \ell\}$, $b_i \in \mathsf{Bd}(\mathsf{P})$ is the unique border element (edge or vertex) containing $\vec{x}_i$.

In other words, the signature of a trajectory is the sequence of border elements (vertices or edges) visited by that trajectory.

Given a finite set S, a simple path in S is a finite sequence $r = e_1e_2 \dots e_k$, for some $k \geq 1$, where for each $i \in \{1, \dots, k\}$, $e_i \in S$ and $e_i \neq e_j$ for $i \neq j$. We let $\mathsf{last}(r) = e_k$ be the last element of r. A *simple cycle* in S is a sequence of the form

$$s = e_0e_1e_2 \dots e_k,$$

for some $k \geq 1$, such that $e_1e_2 \dots e_k$ is a simple path and $e_k = e_0$.

It has been shown in [6] that for each point $\vec{x}_0$, the signature of the trajectory of size ℓ rooted at a point $\vec{x}_0$ can be decomposed, in a *unique way*, as a sequence of the form

$$\sigma^{\ell}_{\vec{x}_0} = r_1 s_1^{k_1} r_2 \dots r_n s_n^{k_n} r_{n+1},$$

where for each $i \in \{1, \dots, n+1\}$, r_i is a simple path, and for each $i \in \{1, \dots, n\}$, $k_i \in \mathbb{N}$, the concatenation $\mathsf{last}(r_i)s_i$ is a simple cycle. The *signature type* of $\sigma^{\ell}_{\vec{x}_0}$ is defined as the sequence

$$\mathsf{type}^{\ell}_{\vec{x}_0} = r_1 s_1 r_2 s_2 \dots r_n s_n r_{n+1}.$$

Even though a 2-PCD system H may admit infinitely many signatures corresponding to infinitely many trajectories, one can show that the number of signature types occurring in such a 2-PCD system is always finite.

Theorem 2.6 ([6]). *For each 2-PCD system $\mathsf{H} = (\mathsf{P}, \varphi, \psi)$, there is only a finite number of signature types corresponding to trajectories in* H.

We note that Theorem 2.6 will be used crucially in the analysis of the behaviour of trajectories in PCD systems on closed surfaces.

2.3 Mortality and Reachability Problems

Given a 2-PCD system $\mathsf{H} = (\mathsf{P}, \varphi, \psi)$, we say that H is *immortal* if there is at least one point $\vec{x}_0 \in \mathsf{bd}(\mathsf{P})$ such that the trajectory $\tau_{\vec{x}_0}$ rooted at $\vec{x}_0$ is infinite[2]. Otherwise, if no such point $\vec{x}_0$ exists, we say that H is *mortal.*

In this work, we will be interested in the following problems for 2-PCD systems and related dynamical systems.

Definition 2.7 (Mortality and Reachability Problems).

Mortality: Given a 2-PCD system $\mathsf{H} = (\mathsf{P}, \varphi, \psi)$, determine whether H is mortal.

Point-to-Point Reachability: Given a 2-PCD system $\mathsf{H} = (\mathsf{P}, \varphi, \psi)$, and points $\vec{x}_0, \vec{x}_f \in \mathsf{bd}(\mathsf{P})$, determine whether $\vec{x}_f$ is reachable from $\vec{x}_0$.

Edge-to-Edge Reachability: Given a 2-PCD system $\mathsf{H} = (\mathsf{P}, \varphi, \psi)$, an initial edge $e_0 \in \mathsf{E}(\mathsf{P})$ and a final edge $e_f \in \mathsf{E}(p)$, determine whether some point $\vec{x}_f \in e_f$ is reachable from some point in $\vec{x}_0 \in e_0$.

2.4 Piecewise Constant Derivative Systems on Surfaces

A surface (or a two-dimensional manifold) S is a compact triangulable space for which in addition the following holds:

(1) Each edge is identified with exactly one other edge;
(2) The triangles identified at each vertex can always be arranged in a cycle $T_1, T_2, \dots, T_k, T_1$ so that adjacent triangles are identified along an edge.

Typical examples of closed surfaces are the sphere, the torus, the connected sum of tori, the Klein bottle and the projective plane. Typical examples of surfaces with boundary are the cylinder and the Möbius strip.

From the computational point of view, we assume that triangulated surfaces are specified by a list of vertices, whose coordinates are assumed to be rational numbers, a list of triples of vertices, defining the triangles, and a list of pairs of identified edges.

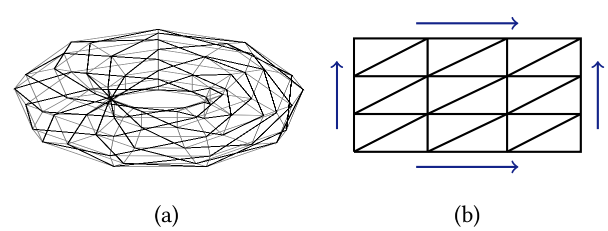


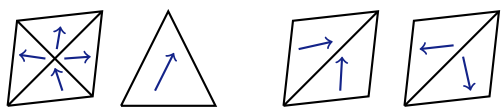
Figure 2: Representation of a triangulated torus: (a) in $\mathbb{R}^3$; (b) as a surface

[2] By an infinite trajectory we mean a trajectory that has infinite size.

A PCD H = (P, φ, ψ) on a closed triangulated surface S is defined analogously to the definition of a PCD on a plane. Following the terminology used in [4], we call such systems PCD_{2m} systems.

In this work, we only consider systems with deterministic behaviour. This is achieved by restricting certain dynamics. We follow the convention used in [4], and require that each vertex can only be the input of at most one region, and that the flow vector on each region cannot be parallel to one of its edges (Figure 3(a)).

We will also need to the definition of a *regular* PCD_{2m} as introduced in [20]. These systems forbid two additional types of dynamics when compared with general PCD_{2m} systems. (Figure 3(b)).



(a) (b)

Figure 3: (a) Two types of dynamics forbidden in PCD_{2m}: branching at a vertex and flow vector parallel to an edge; (b) Two types of dynamics forbidden in PCD_{r2m}

Note that the above restrictions mean that each region (triangle) has either two input edges and one output edge (a two-to-one region); or one input edge and two output edges (a one-to-two region) (Figure 4).

(a) (b)

Figure 4: (a) Two input edges and one output edge; (b) and one input edge and two output edges

Definition 2.8 (PCD_{r2m}). We say that a PCD_{2m} system H = (P, φ, ψ) is regular if for any $p, p' \in \mathsf{P}$ such that $\mathsf{Bd}(p) \cap \mathsf{Bd}(p') \neq \emptyset$ exactly one of the following holds for any $b \in \mathsf{Bd}(p) \cap \mathsf{Bd}(p')$:

(1) $b \in \mathsf{In}(p)$ and $b \in \mathsf{Out}(p')$;
(2) $b \in \mathsf{Out}(p)$ and $b \in \mathsf{In}(p')$.

For brevity, we refer to regular PCD_{2m} systems as PCD_{r2m} *systems.* We note that all trajectories in PCD_{r2m} *systems* are immortal. Nevertheless by analysing the behaviour of these immortal trajectories in PCD_{r2m} systems we will be able to infer properties of trajectories in general PCD_{2m} systems.

3 DECIDING MORTALITY FOR PCD SYSTEMS ON SURFACES

In this section, we will prove the first result of this work (Theorem 3.1), which states that the mortality problem for PCD_{2m} systems is decidable.

Our proof will follow from a reformulation of the problem of determining the *existence* of immortal trajectories in such a system H into a *case analysis about the behaviour of trajectories* in a suitable PCD_{r2m} system H′. We note that all trajectories in H′ are guaranteed to be immortal. Nevertheless, the behaviour of each such an immortal trajectory in H′ carries information about whether it originates from some *mortal* or some *immortal* trajectory in the original PCD_{2m} system H.

THEOREM 3.1. *The mortality problem is decidable for* PCD_{2m} *systems.*

We dedicate the remainder of this section to the proof of Theorem 3.1 which consists of several steps, where each step is dealt with in a separate subsection. A road-map for these steps is as follows.

(1) In Section 3.1 we state a theorem that classifies regions in the underlying manifold of a *regular* piecewise constant derivative system into three types according to the behaviour of trajectories in the respective region. The theorem is a special case of the results due to Aranson [3] and Mayer [16].
(2) In Section 3.2 we provide the definition of Rauzy graphs and show how they can be used to analyse the behaviour of regular PCD systems on closed surfaces.
(3) In Section 3.3 we introduce a new technique to transform a given PCD_{2m} system H into a PCD_{r2m} H'. It is important to note that while all trajectories in H' are immortal, the behaviour of each such trajectory can be used to determine whether it arises from a mortal or an immortal trajectory in the original system H.
(4) In Section 3.5 we finally provide a proof of Theorem 3.1.

3.1 Trajectories on Closed Surfaces

A closed surface can be decomposed under certain conditions into maximal connected components consisting of trajectories with similar behaviour [3, 16]. Before stating a formal decomposition theorem, we define three types of trajectory: *closed*, *dense*, and *orbital stable*.

Definition 3.2 (Closed trajectory). A trajectory $\tau_{\vec{x}_0} = \vec{x}_0\vec{x}_1\vec{x}_2\ldots$ is called *closed* if there is an $i > 0$ such that $\vec{x}_0 = \vec{x}_i$.

Let e be an edge of a piecewise constant derivative system. A sub-edge of e is a connected subset of e that is not a singleton point.

Definition 3.3 (Dense trajectory). A trajectory $\tau_{\vec{x}_0}$ is called *dense* on a set of sub-edges $e_1, \ldots, e_k$ if for each $i \in \{1, \ldots, k\}$, and each $e_i' \subseteq e_i$, there is some $\vec{x}_0' \in e_i'$ such that $\vec{x}_0'$ is reachable from $\vec{x}_0$.

Below, we give a formal definition of the notion of an orbital stable trajectory. Intuitively, a trajectory $\tau_{\vec{x}_0}$ is orbital stable if each trajectory $\tau_{\vec{y}_0}$ initiating closely enough to $\tau_{\vec{x}_0}$ remains close to $\tau_{\vec{x}_0}$.

Definition 3.4 (Orbital stable trajectory). A trajectory $\tau_{\vec{x}_0}$ is called *orbital stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each $\vec{y}_0$ in the δ-neighbourhood of $\tau_{\vec{x}_0}$, the trajectory $\tau_{\vec{y}_0}$ is contained in the ε-neighbourhood of $\tau_{\vec{x}_0}$.

A classical result from Mayer states that a closed orientable surface of arbitrary genus can be decomposed into components, where trajectories are either closed, or orbital stable, or dense (Theorem IX of [16]). Furthermore, these regions are separated by closed trajectories called *separatrices.* This result was extended by Aranson to the context of non-orientable manifolds (Theorem 4 of [3]). For completeness, we state both theorems below as Theorem 3.5 and Theorem 3.6 respectively. Subsequently, we state a combined version of these theorems restricted to the context of PCD_{r2m}(Theorem 3.7).

Mayer's and Aranson's theorems are stated in a more general context, where a dynamical system on a manifold M is defined by a finite cover $\{r_1, \ldots, r_m\}$ of M, where each r_i is a region homeomorphic to the unit disc. Each such region r_i has its own coordinate system, (u_i, v_i), and the dynamics each region is specified by differential equations $du_i/dt = U(u_i, v_i)$ and $dv_i/dt = V(u_i, v_i)$.

Below, we let S_g be a closed 2-dimensional manifold (surface) with genus $g \geqslant 0$, and R be a covering of this manifold by a finite number of regions r_i, $1 \leqslant i \leqslant n$, homeomorphic to an Euclidean disc such that every region r_i has its own coordinate system (u_i, v_i). Additionally, we let the dynamics in each r_i be defined by differential equations

$$\frac{du_i}{dt} = U(u_i, v_i) \quad \text{and} \quad \frac{dv_i}{dt} = V(u_i, v_i) \tag{1}$$

satisfying the following properties:

(a) The right-hand sides of the system of equations (1) are functions of class C^1 (or more precisely, C^{r-1} for $r \geq 2$) or of analytic class. That is the functions must be differentiable, and hence continuous.
(b) At points common to two or more areas, the transition from one system of equations to another system of equations is accomplished by transforming one coordinate system into another coordinate system.
Transition from one coordinate system to another at common points is accomplished by a function of class C^1 or of analytic class with nonzero Jacobian.

Theorem 3.5 (for orientable manifolds) and Theorem 3.6 (for non-orientable manifolds) below show that any two-dimensional manifold can be decomposed into dynamical components containing trajectories which are equivalent topologically.

Theorem 3.5 (Mayer, [16]). *Let S_g be a dynamical system on a closed two-dimensional orientable manifold of genus g. Then S_g is a disjoint union of a finite number of areas $M_1, \ldots, M_k$ (referred later as* ***dynamical components****) of the following types:*

(1) ***Type A****: Any trajectory inside the area is orbital stable and non-closed. Furthermore, all the trajectories have the same set of limit points; the area is flat and at most 2-connected;*
(2) ***Type B****: Any trajectory inside the area is closed; the area is either flat and at most 2-connected or equals to the whole manifold in case of $g = 1$ (only for a torus);*
(3) ***Type C****: Any trajectory inside the area is everywhere dense; the area is not flat and the number of areas of this type does not exceed g.*

All other trajectories, called ***separatrices****, form boundaries between the areas of the above types.*

Theorem 3.6 (Aranson, [3]). *Let S_g be a dynamical system on a closed two-dimensional non-orientable manifold of genus g. Then S_g is a disjoint union of a finite number of areas $M_1, \ldots, M_k$, each of them having one of the following types:*

(1) ***Type A****: Any trajectory inside the area is orbital stable and non-closed. Furthermore, all the trajectories have the same set of limit points; the area is flat and at most 2-connected;*
(2) ***Type B****: Any trajectory inside the area is closed; the area is either flat and at most 2-connected or is homeomorphic to Möbius strip or equals to the whole manifold (Kleine bottle) in case of $g = 2$;*
(3) ***Type C****: Any trajectory inside the area is everywhere dense; the area is not flat and not homeomorphic to Mobius strip. The number of areas of this type does not exceed $\left[\frac{g-1}{2}\right]$.*

All other trajectories form boundaries between the areas of the above types.

Both, Mayer's and Aranson's theorems, hold for a much wider class of dynamical systems than the ones we consider in this work. Indeed, PCD_{r2m} systems straightforwardly satisfy Conditions (a) and (b) for both types of surfaces, orientable and non-orientable, as justified below.

(a) Condition (a) is satisfied for both types of surfaces, orientable and non-orientable, since by definition of PCD_{r2m} the trajectory inside each region is linear, and therefore the derivatives du_i/dt and dv_i/dt in Equation (1) are constants, and therefore continuous in their domains (regions).
(b) Condition (b) is satisfied for both types of surfaces, orientable and non-orientable, because the following holds.
   - The manifold is covered by planar regions whose closures may intersect either on an interval or on a point.
   - For each of these planar regions, we may use a Cartesian coordinate system by choosing two perpendicular lines and by setting the coordinate of each point as the pair of distances to these lines.
     In this way, if two regions intersect in an interval or in a point, then the conversion from the coordinate system of one of the regions to the coordinate system of the other region can be done by a linear function, and therefore continuous with non-zero Jacobian.

Note that due to possible sharp edges on its surface, a PCD_{r2m} system is not necessary a smooth manifold. Nevertheless, we can still consider such systems as a special case of the systems defined by Mayer and Aranson as the requirement here for each function is to be continuous in its domain. Besides, edges can be smoothed without affecting the behaviour of trajectories as in dimensions up to three, topological and smooth manifolds coincide [2, 23].

Now given the above, Theorems 3.5 and Theorem 3.6 can be restated in the context of PCD_{r2m} in the following simplified way.

Theorem 3.7. *A PCD_{r2m} on a closed surface of genus g, is a disjoint union of a finite number of areas $C_1, \ldots, C_k$, each of which has one of the following types:*

(1) ***Type A****: Any trajectory inside the area is orbital stable and non-closed. Furthermore, the area is planar.*
(2) ***Type B****: Any trajectory inside the area is closed.*
(3) ***Type C****: Any trajectory inside the area is everywhere dense.*

All other trajectories, are called ***separatrices****. These are closed trajectories that form boundaries between the areas of the above types.*

It is interesting to note that intuitively, a Type A area is equivalent to a 2-PCD system (on a plane), for which point-to-point and edge-to-edge reachability are known to be decidable [15]. We note that since the number of regions $C_1, ..., C_k$ in a PCD_{r2m} system is finite, the number of separatrices is also finite.

3.2 Rauzy Graphs and Regular PCD Systems

In this section, we recall the classic definition of a Rauzy graph of power k of a *factorial prolongable* formal language [18].

An *alphabet* is any finite non-empty set A. We may refer to the elements of such an alphabet as *letters*. We let A^* denote the set of all finite words constructed with letters from A. Given a word $w \in \mathsf{A}^*$, we let $\mathrm{Letters}(w)$ be the set of letters occurring in w. A *language* over A is a subset $\mathsf{L} \subseteq \mathsf{A}^*$. Below, we define the notions of factorial and prolongable languages.

Definition 3.8 (Factorial language). A language L over an alphabet A is *factorial* if $u_0u_1 \dots u_n \in \mathsf{L}$ implies $u_1u_2 \dots u_n \in \mathsf{L}$ and $u_0u_1 \dots u_{n-1} \in \mathsf{L}$ for arbitrary $u_0, \dots, u_n \in \mathsf{A}$.

Definition 3.9 (Prolongable language). A language L over an alphabet A is *prolongable* if for any $u \in \mathsf{L}$ there exist $a, b \in \mathsf{A}$ such that $au \in \mathsf{L}$ and $ub \in \mathsf{L}$.

Given a factorial prolongable language L over A and a positive integer $k \in \mathbb{N}$, the k-th power Rauzy graph for L is the directed graph whose vertices are the words of L of size k, and whose arcs are pairs (u, v) of strings of size k with the property that there exists a word w in L of size $k + 1$ which has u as a prefix and v as a suffix. A more precise definition is given below.

Definition 3.10 (Rauzy graph). Let L be a factorial and prolongable language over an alphabet A, and let $k \geq 1$. The k-th power Rauzy graph for L is the directed graph $\mathsf{R}^k(\mathsf{L}) = (\mathsf{V}^k, \mathsf{E}^k)$ defined as follows:

(1) $\mathsf{V}^k = \{w \in \mathsf{L} \mid |w| = k\}$;
(2) For any vertices $u = u_1u_2 \dots u_k \in \mathsf{V}^k$ and $v = v_1v_2 \dots v_k \in \mathsf{V}^k$ there is an edge $(u, v) \in \mathsf{E}^k$ if and only if $u_2 = v_1, u_3 = v_2, \dots, u_k = v_{k-1}$ and $u_1u_2 \dots u_kv_k \in \mathsf{L}$.

Given a 2-PCD_{2m} system $\mathsf{H} = (\mathsf{P}, \varphi, \psi)$, we let $\mathsf{L}(\mathsf{H})$ be the set of signatures of finite trajectories in H. We note that $\mathsf{L}(\mathsf{H})$ is the language of H over the alphabet $\mathsf{Bd}(\mathsf{P})$.

Proposition 3.11. *Let* $\mathsf{H} = (\mathsf{P}, \varphi, \psi)$ *be a* PCD_{r2m} *system. Then* $\mathsf{L}(\mathsf{H})$ *is a factorial and prolongable language over the alphabet* $\mathsf{Bd}(\mathsf{P})$.

Proof. Let $\tau^r_{\vec{x}_0} = \vec{x}_0\vec{x}_1 ... \vec{x}_r$ be a trajectory in H. Since H is a PCD_{r2m} system, there are points $\vec{x}_0'$ and $\vec{x}_{r+1}$ such that both $\vec{x}_0'\tau^r_{\vec{x}_0}$ and $\tau^r_{\vec{x}_0}\vec{x}_{r+1}$ are trajectories in H. Now, we also have that $\tau^{r-1}_{\vec{x}_0}$ is a prefix of $\tau^r_{\vec{x}_0}$ and that $\tau^{r-1}_{\vec{x}_1}$ is a suffix of $\tau^r_{\vec{x}_0}$. This shows that $\mathsf{L}(\mathsf{H})$ is prolongable and factorial. □

Since the language of $\mathsf{L}(\mathsf{H})$ of any PCD_{r2m} system H is factorial and prolongable, for each $k \geq 1$, the k-th power Rauzy graph of $\mathsf{L}(\mathsf{H})$ is well defined. The following lemma describes some structural properties of Rauzy graphs associated with a PCD_{r2m} system. The dynamical cells mentioned in Lemma 3.12 are the same ones mentioned in Theorem 3.7.

Lemma 3.12. *Let* $\mathsf{H} = (\mathsf{P}, \varphi, \psi)$ *be a* PCD_{r2m} *system with n dynamic cells, and let* L *be the language of* H *over the alphabet* $\mathsf{A} = \mathsf{Bd}(\mathsf{P})$. *Then there is some* $s \in \mathbb{N}$ *such that for each* $i \geqslant s$, *the Rauzy graph* $\mathsf{R}^i(\mathsf{L})$ *consists of* $k \geqslant n$ *disconnected components* $K^i_1 = (\mathsf{V}^i_1, \mathsf{E}^i_1), \dots, K^i_k = (\mathsf{V}^i_k, \mathsf{E}^i_k)$ *such that for each* $j \in \{1, \dots, k\}$, *at least one of the following conditions holds for each component* $K^i_j = (\mathsf{V}^i_j, \mathsf{E}^i_j)$, $1 \leqslant j \leqslant k$:

(1) *There is a subset* $S \subseteq \mathsf{A}$ *such that each word* $w \in \mathsf{V}^i_j$ *consists of all letters of* S.
(2) *There exists* s', *with* $i \geqslant s > s'$, *such that the set of signature types of words in* V^i_j *equals the set of signature types of words in some component of the graph* $\mathsf{R}^{s'}(\mathsf{L})$.

Proof sketch. Intuitively, each component of the Rauzy graph $\mathsf{R}^i(\mathsf{L})$ corresponds to a dynamic cell as specified in Theorem 3.7. A component satisfying Condition 1 of the lemma corresponds to a cell of type B or C, while a component satisfying Condition 2 corresponds to a dynamic cell of type A.

If the component corresponds to a dynamic cell of type B, then every trajectory in this cell is closed, and therefore the trajectory is periodic. This means that each edge occurring in this trajectory will reach each other letter occurring in the trajectory.

If V^i_j corresponds to a dynamic cell of type C, then trajectories are dense, and therefore each letter will reach each other letter in at most r steps for some finite r (Lemma 1 on the *uniformity condition* in [20]).

Finally, if a component corresponds to a cell of type A, then this dynamic cell is homeomorphic to the disc. Therefore, by Theorem 2.6 the set of signature types of trajectories in this region is finite. □

We note that Lemma 3.12 is a generalisation of the lemma that was proved in [20] for PCD_{r2m} systems on orientable closed surfaces and based on Mayer's classification of trajectories on orientable closed surfaces [16]. Here we state it for a more general case that includes both orientable and non-orientable closed surfaces if we take into account the results due to Aranson [3].

3.3 Immortalisation of a Trajectory at an Edge

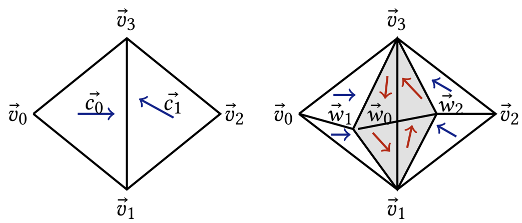
For any point $\vec{x}$ in the polygonal partition of a PCD_{r2m} system, the trajectory $\tau_{\vec{x}}$ is immortal. Additionally, any such trajectory is *left-extendable*, meaning that is always a point $\vec{x}'$ such that $\tau'_{\vec{x}} = \vec{x}'\tau_{\vec{x}}$. In other words, any trajectory can be extended by adding one step at its beginning. On the other hand, in general PCD_{2m} systems, trajectories may be finite, and may be not left-extendable. In this section we introduce a new technique for immortalisation of trajectories of general PCD_{2m} systems.

In order to map trajectories in a PCD_{2m} system that are either finite or not left-extendable to trajectories in a PCD_{r2m} system, we will use the notions of attractors and distractors, which are defined below.

Definition 3.13 (Attractor). Given a PCD_{2m} system $\mathsf{H} = (\mathsf{P}, \varphi, \psi)$, we say that a set of regions $p_1, \dots, p_k \in \mathsf{P}$, where $k \geqslant 3$, is an attractor if for each $\vec{x} \in \bigcup_{i=1}^k \mathrm{bd}(p_i)$ there is an $\vec{x}' \in \bigcup_{i=1}^k \mathrm{bd}(p_i)$ such that $(\vec{x}, \vec{x}')$ is a step.

Definition 3.14 (Distractor). Given a PCD_{2m} system $\mathsf{H} = (\mathsf{P}, \varphi, \psi)$, we say that a set of regions $p_1, \ldots, p_k \in \mathsf{P}$, where $k \geqslant 3$, is a distractor if for each $\vec{x} \in \bigcup_{i=1}^{k} \mathrm{bd}(p_i)$ there is an $\vec{x}' \in \bigcup_{i=1}^{k} \mathrm{bd}(p_i)$ such that $(\vec{x}', \vec{x})$ is a step.

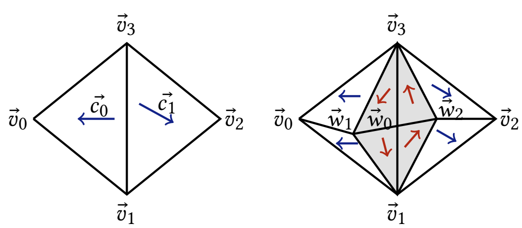
We note that the only reason for the existence of a finite trajectory in a PCD_{2m} system is the existence of a pair of regions p and p' sharing a common boundary edge e that is an output edge of both regions (Figure 5(a)). We say that e is a *clashing edge*.



(a) (b)

Figure 5: (a) The clashing edge $(\vec{v}_1, \vec{v}_3)$; (b) replacing the clashing edge $(\vec{v}_1, \vec{v}_3)$ by an attractor

On the other hand, the only reason for the existence of a trajectory that is not left-extendable is the existence of a pair of regions p and p' having a common boundary edge e such that e is an input edge of both regions (Figure 6(a)). We say that e is a *diverging edge*.



(a) (b)

Figure 6: (a) The diverging edge $(\vec{v}_1, \vec{v}_3)$; (b) replacing the diverging edge $(\vec{v}_1, \vec{v}_3)$ by a distractor

Given a PCD_{2m} system $\mathsf{H} = (\mathsf{P}, \varphi, \psi)$, the immortalisation of H is the system $\mathsf{H}' = (\mathsf{P}', \varphi', \psi')$ obtained from H by replacing each clashing edge by a suitable attractor consisting of four regions, and each diverging edge by a suitable distractor consisting of four regions. We illustrate these constructions in Figure 5(b) and Figure 6(b).

In Figure 5(b), the clashing edge $\vec{v}_1\vec{v}_3$ is a common boundary of the regions $\vec{v}_0\vec{v}_1\vec{v}_3$ and $\vec{v}_2\vec{v}_1\vec{v}_3$. The edge $\vec{v}_1\vec{v}_3$ is then replaced by an attractor consisting of four regions $\triangle\vec{w}_1\vec{v}_1\vec{w}_0$, $\triangle\vec{w}_0\vec{v}_1\vec{w}_2$, $\triangle\vec{w}_0\vec{w}_2\vec{v}_3$, and $\triangle\vec{w}_1\vec{w}_0\vec{v}_3$ with dynamics of type two-to-one, meaning that in each of these four regions there are two input edges and one output edge.

In Figure 6(b), the diverging edge $\vec{v}_1\vec{v}_3$ is a common boundary of the regions $\vec{v}_0\vec{v}_1\vec{v}_3$ and $\vec{v}_2\vec{v}_1\vec{v}_3$. The edge $\vec{v}_1\vec{v}_3$ is then replaced by a distractor consisting of four regions $\triangle\vec{w}_1\vec{v}_1\vec{w}_0$, $\triangle\vec{w}_0\vec{v}_1\vec{w}_2$, $\triangle\vec{w}_0\vec{w}_2\vec{v}_3$, and $\triangle\vec{w}_1\vec{w}_0\vec{v}_3$ with dynamics of type one-to-two, meaning that in each of these four regions there is one input edge and two output edges.

A more formal construction of H' is given below. Let $\mathsf{R}^{\mathsf{ad}}(\mathsf{H}) = \mathsf{R}^{\mathsf{a}}(\mathsf{H}) \cup \mathsf{R}^{\mathsf{d}}(\mathsf{H})$, where the sets $\mathsf{R}^{\mathsf{a}}(\mathsf{H}) \subseteq \mathsf{P} \times \mathsf{P}$ and $\mathsf{R}^{\mathsf{d}}(\mathsf{H}) \subseteq \mathsf{P} \times \mathsf{P}$ are defined as follows:

(1) $\mathsf{R}^{\mathsf{a}}(\mathsf{H}) = \{(p, p') \in \mathsf{P} \times \mathsf{P} \mid \exists e \in \mathsf{E}(\mathsf{P}) : e \in \mathsf{Out}(p) \cap \mathsf{Out}(p')\}$; and
(2) $\mathsf{R}^{\mathsf{d}}(\mathsf{H}) = \{(p, p') \in \mathsf{P} \times \mathsf{P} \mid \exists e \in \mathsf{E}(\mathsf{P}) : e \in \mathsf{In}(p) \cap \mathsf{In}(p')\}$.

Let $(p_1, p_2) \in \mathsf{R}^{\mathsf{ad}}(\mathsf{H})$ be a pair of regions with dynamics defined by vectors $\varphi(p_1) = \vec{c}_1$ and $\varphi(p_2) = \vec{c}_2$, such as depicted in Figure 6(a). For simplicity, we let $p_1 = \triangle\vec{v}_0\vec{v}_1\vec{v}_2$ be the region defined by the vertices $\vec{v}_0$, $\vec{v}_1$, and $\vec{v}_3$; and $p_2 = \triangle\vec{v}_1\vec{v}_2\vec{v}_3$ be the triangle defined by the vertices $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$. To construct P', we subpartition each such pair $(p_1, p_2) \in \mathsf{R}^{\mathsf{ad}}$ by adding a quadrilateral $\vec{v}_1\vec{w}_1\vec{v}_3\vec{w}_2$ such that

(1) $\vec{w}_1 \in p_1$ and $\vec{w}_2 \in p_2$;
(2) The segment between $\vec{w}_1$ and $\vec{w}_2$ is not parallel to any of the vectors $\vec{c}_1$ and $\vec{c}_2$;
(3) $\vec{w}_0$ is the intersection of the segments between $\vec{v}_1$ and $\vec{v}_3$ and between $\vec{w}_1$ and $\vec{w}_2$.

We define the function φ' assigning the vector associated with each region in P' as follows:

(1) If for $p' \in \mathsf{P}'$ there is $p \in \mathsf{P}$ such that $p' = p$ then $\varphi'(p') = \varphi(p)$.
(2) If $(p_1, p_2) \in \mathsf{R}^{\mathsf{a}}$, we create an attractor by assigning to all four regions $\triangle\vec{w}_1\vec{v}_1\vec{w}_0$, $\triangle\vec{w}_0\vec{v}_1\vec{w}_2$, $\triangle\vec{w}_0\vec{w}_2\vec{v}_3$, and $\triangle\vec{w}_1\vec{w}_0\vec{v}_3$ the dynamics of type two-to-one to ensure that every trajectory is spiralling in, towards the point $\vec{w}_0$ and therefore stays inside the union of the above regions.
(3) If $(p_1, p_2) \in \mathsf{R}^{\mathsf{d}}$, we create a distractor by assigning to all four regions $\triangle\vec{w}_1\vec{v}_1\vec{w}_0$, $\triangle\vec{w}_0\vec{v}_1\vec{w}_2$, $\triangle\vec{w}_0\vec{w}_2\vec{v}_3$, and $\triangle\vec{w}_1\vec{w}_0\vec{v}_3$ the dynamics of type one-to-two to ensure that every trajectory is spiralling out and eventually leaves the union of the above regions.
(4) To each of the new regions $p'_1 = \triangle\vec{v}_0\vec{v}_1\vec{w}_1$, $p'_2 = \triangle\vec{v}_1\vec{v}_2\vec{w}_2$, $p'_3 = \triangle\vec{w}_2\vec{v}_2\vec{v}_3$, $p'_4 = \triangle\vec{v}_0\vec{w}_1\vec{v}_3$, we assign the same vector field as in the region of P it is a subset of.

To define the function ψ', we assign the new border elements $\vec{w}_1$, $\vec{w}_2$, $e'_1 = (\vec{v}_1, \vec{w}_1)$, $e'_2 = (\vec{v}_1, \vec{w}_2)$, $e'_3 = (\vec{w}_2, \vec{v}_3)$, $e'_4 = (\vec{w}_1, \vec{v}_3)$ to the regions as follows:

(1) We assign the new vertex $\vec{w}_1$ to any of the regions p'_1, p'_4, and $\vec{w}_2$ to any of the regions p'_2, p'_3.
(2) $\psi'(e'_i) = p'_i$, $1 \leqslant i \leqslant 4$.

By construction, $\mathsf{R}^{\mathsf{ad}}(\mathsf{H}') = \emptyset$, and, hence, H' is a PCD_{r2m}.

3.4 Immortalisation of a Trajectory at a Boundary

Now we extend the technique introduced in Section 3.3 to define the immortalisation of trajectories at a boundary (for surfaces with boundary).

Given a PCD_{2m} system $\mathsf{H} = (\mathsf{P}, \varphi, \psi)$, the immortalisation of H is the system $\mathsf{H}' = (\mathsf{P}', \varphi', \psi')$ obtained from H by replacing each boundary by a set of attractors and distractors as described above. We illustrate this construction in Figure 7.

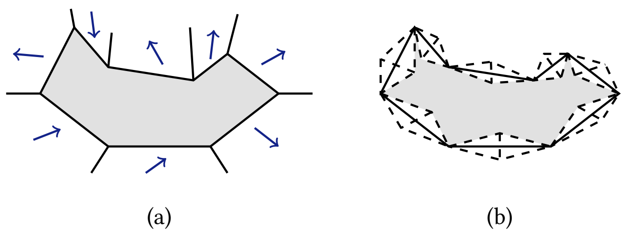


Figure 7: (a) A boundary; (b) the boundary replaced by a set of attractors and distructors

The rest of the 'interior' region of the border is homeomorphic to the disc, and it can be replaced by a 2-PCD system H′ with trajectories of type A as defined in Theorem 3.7. By construction, H′ is a PCD_{r2m}.

It is worth mentioning here that the cylinder is equivalent to a sphere with two disks removed, while the Möbius strip can be seen as a projective plane with a disk removed.

3.5 Proof of Theorem 3.1

Finally, we are in the position to prove our first result on decidability of mortality for PCD_{2m} systems stated in Theorem 3.1.

Proof of Theorem 3.1. Let $\mathsf{H} = (\mathsf{P}, \varphi, \psi)$ be a PCD_{2m} system and $\mathsf{H}' = (\mathsf{P}', \varphi', \psi')$ be the immortalisation of H obtained according to the constructions given in Section 3.3 and Section 3.4. Note that H′ is a PCD_{r2m}.

Let L be the language of H over the alphabet $\mathsf{A} = \mathsf{Bd}(\mathsf{P})$, and L′ be the language of H′ over the alphabet $\mathsf{A}' = \mathsf{Bd}(\mathsf{P}')$. Then by Lemma 3.12, there is some $s \in \mathbb{N}$, and some $k \geqslant n$ such that the Rauzy graph $\mathsf{R}^s(\mathsf{L}')$ consists of strongly connected components $\{K_j^s = (\mathsf{V}_j^s, \mathsf{E}_j^s)\}_{j \in \{1,\dots,k\}}$ such that at least one of the following conditions hold for each component K_j:

1. There is a subset $S \subseteq \mathsf{A}'$ such that for each word $w \in \mathsf{V}_j^s$, $\mathsf{Letters}(w) = S$.
2. There is an $s' < s$ such that the set of signature types of V_j^s is equal to the set of signature types of the vertices of some component of $\mathsf{R}^{s'}(\mathsf{L}')$.

Let n be the number of dynamic cells of H′. Our reasoning below takes into consideration the following observations:

(a) Attractors in H′ give rise to orbital stable trajectories converging to a point. In Figure 6(a), the point of convergence is illustrated by $\vec{w}_0$. Note that such a trajectory evolves in a spiral, always getting closer to $\vec{w}_0$ but never actually reaching it. These trajectories correspond to the halting trajectories in H.

(b) For each point $\vec{x}$, $\tau_{\vec{x}}$ is closed (dense respectively) and has type T in H if and only if $\tau_{\vec{x}}$ is closed (dense respectively) and has type T in H′.

Let k_A, k_B, and k_C be the number of components in $\mathsf{R}^s(\mathsf{L}')$ corresponding to dynamic cells of types A, B and C respectively, and k_S be the number of components in $\mathsf{R}^s(\mathsf{L}')$ corresponding to separatrices. Then $k_A + k_B + k_C + k_S = k$, where $k > 0$. Now we consider the following cases:

(i) Let $k_B > 0$ or $k_S > 0$. Then there exists a closed trajectory $\tau'_{\vec{x}'_0}$ in H′, and a closed trajectory $\tau_{\vec{x}_0}$ in H such that $\mathrm{type}'_{\vec{x}'_0} = \mathrm{type}_{\vec{x}_0}$ with $\vec{x}'_0 = \vec{x}_0$.
All closed trajectories are trivially immortal. Hence, H is immortal.

(ii) Let $k_C > 0$. Then there is a dense trajectory $\tau'_{\vec{x}'_0}$ in H′. By construction, there is a dense trajectory $\tau_{\vec{x}_0}$ in H such that $\mathrm{type}'_{\vec{x}'_0} = \mathrm{type}_{\vec{x}_0}$ with $\vec{x}'_0 = \vec{x}_0$. All dense trajectories are trivially immortal. Hence, H is immortal.

(iii) Let $k_B = k_C = k_S = 0$. Then $k_A > 0$, and H′ has dynamic cells $C_1^A, \dots, C_{k_A}^A$ of type A with orbital stable trajectories. By Theorem 3.7, each of these dynamic cells is planar, and, hence, it is a 2-PCD. Then by Theorem 2.6 there is a finite number r of signature types corresponding to the trajectories in all these components. By Corollary 5 in [15], the signature type of such a trajectory has the form $\sigma^i = b_0^i \dots b_{l_i}^i (b_{l_i+1}^i \dots b_{l_i+m_i}^i)^\omega$.
Let $\Sigma^a \subseteq \Sigma'$ be the alphabet of attractors. If for any signature type σ^i over Σ^a, there is $b_j^i \in \sigma^i$ such that $b_j^i \in \Sigma^a$ then each corresponding trajectory of H halts, and, therefore, H is mortal. Otherwise it is immortal.

□

4 DECIDING EDGE-TO-EDGE REACHABILITY FOR PCD SYSTEMS ON SURFACES

It has been shown in [20] that edge-to-edge reachability is decidable for *regular* piecewise constant derivative systems on closed *orientable* two-dimensional manifolds.

Theorem 4.1 ([20]). *The edge-to-edge reachability problem is decidable for* PCD_{r2m} *systems.*

In this section we show that edge-to-edge reachability is decidable for PCD_{2m} systems by removing both the regularity restriction on PCD systems on surfaces on the one hand and the requirements on the manifolds such as boundary and orientation on the other hand. This solves the edge-to-edge reachability version of an open problem from [4].

Theorem 4.2. *The edge-to-edge reachability problem is decidable for* PCD_{2m} *systems.*

Proof. Let $\mathsf{H} = (\mathsf{P}, \varphi, \psi)$ be a PCD_{2m} and the reachability task is given by the edges $e_0, e_f \in \mathsf{E}(\mathsf{P})$, where e_0 is the initial edge and e_f is the final edge. Let PCD_{r2m} $\mathsf{H}' = (\mathsf{P}', \varphi', \psi')$ be the immortalisation of H as defined in Section 3.3.

We define edges e_0^i, e_f^j, where $i \in \{1, 2\}$ and $j \in \{1, 4\}$ (as depicted in Figure 8). Then e_f is reachable from e_0 in H if and only if e_f^j is reachable from e_0^i in H′ for some i, j.

Since from Theorem 4.1, edge-to-edge reachability is decidable for PCD_{r2m} systems, our reduction implies that edge-to-edge reachability is decidable in for PCD_{2m} systems as well.

We use below the sets $\mathsf{R^a}(\mathsf{H}) \subseteq \mathsf{P}\times\mathsf{P}$ and $\mathsf{R^d}(\mathsf{H}) \subseteq \mathsf{P}\times\mathsf{P}$ defined in Section 3.3 as follows:

(1) $\mathsf{R^a}(\mathsf{H}) = \{(p,p') \in \mathsf{P}\times\mathsf{P} \mid \exists e \in \mathsf{E}(\mathsf{P}) : e \in \mathsf{Out}(p) \cap \mathsf{Out}(p')\}$; and
(2) $\mathsf{R^d}(\mathsf{H}) = \{(p,p') \in \mathsf{P}\times\mathsf{P} \mid \exists e \in \mathsf{E}(\mathsf{P}) : e \in \mathsf{In}(p) \cap \mathsf{In}(p')\}$.

Now we only need to consider the cases $(p_1,p_2) \in \mathsf{R^a}(\mathsf{H}) \cup \mathsf{R^d}(\mathsf{H})$.

Let $e = (\vec{v}_1,\vec{v}_3) \in \mathsf{E}(\mathsf{P})$ be either the initial edge e_0 or the final edge e_f, and let $(\vec{v}_1,\vec{v}_3) = \mathsf{cl}(p_1) \cap \mathsf{cl}(p_2)$ for some regions p_1,p_2.

(1) Let $(p_1,p_2) \in \mathsf{R^a}$. If $e_f = (\vec{v}_1,\vec{v}_3)$ then the reachability task in H is replaced by four reachability tasks in H' with the final edges are $e_f^1, e_f^2, e_f^3, e_f^4 \in \mathsf{E}(\mathsf{P}')$. See Figure 8(a) for an example.
(2) Let $(p_1,p_2) \in \mathsf{R^d}$. If $e_0 = (\vec{v}_1,\vec{v}_3) \in \mathsf{E}(p_1)$ then we replace the reachability task for H with two reachability tasks with the initial edges e_0^1, e_0^2 (depicted with the solid line in Figure 8(b)). If $e_0 = (\vec{v}_1,\vec{v}_3) \in \mathsf{E}(p_2)$ then we replace the reachability task for H with two reachability tasks with the initial edges e_0^2, e_0^3 (depicted with the dotted line in Figure 8(b)).

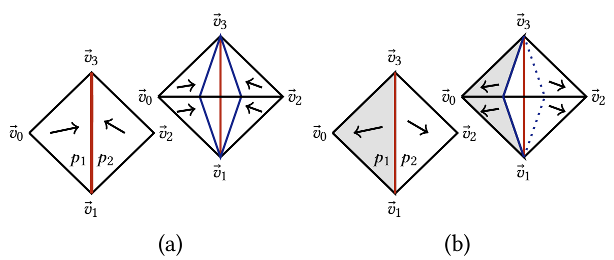


Figure 8: A reachability task for $\mathsf{H}' = (\mathsf{P}',\varphi')$: (a) $e_f = (\vec{v}_1,\vec{v}_3)$; (b) $e_0 = (\vec{v}_1,\vec{v}_3) \in \mathsf{E}(p_1)$

We deal with attractors and destructors at boundary (for surfaces with boundary) in a similar way.

Intuitively, in order to check reachability on H', it is sufficient to build a finite sequence $R^1(L'),\ldots,R^{t_{stop}}(L')$ of Rauzy graphs, where L' is the language of H' and t_{stop} is defined by Lemma 3.12. An edge $e_f \in \mathsf{E}'(\mathsf{P})$ is reachable from an edge $e_0 \in \mathsf{E}'(\mathsf{P})$ if and only if $R^{t_{stop}}(L')$ contains a component with a vertex labelled by a word $(\ldots e_0 \ldots e_f \ldots)$.

By Theorem 4.1, edge-to-edge reachability is decidable for H' as it is a PCD_{r2m}. It is also decidable for a PCD_{2m} H by construction. □

5 DECIDING INTERVAL-TO-INTERVAL REACHABILITY FOR REGULAR PAM

Decidability of reachability for one-dimensional piecewise affine maps (PAMs) is a long-standing open problem, even for the case when it is made of only two intervals [12]. In this section we extend our results further and show that the interval-to-interval version of the reachability problem for the class of regular one-dimensional PAMs is also decidable.

A rational interval is a subset of $\mathbb{R}$ of one of the following forms: $[\vec{x},\vec{y}]$, $[\vec{x},\vec{y})$, $(\vec{x},\vec{y}]$, $(\vec{x},\vec{y})$, $(-\infty,\vec{y}]$, $(-\infty,\vec{y})$, $[\vec{x},\infty)$, $(\vec{x},\infty)$, where $\vec{x},\vec{y} \in \mathbb{Q}$ such that $\vec{x} \le \vec{y}$.

Definition 5.1 (PAM). Let I_i be a finite set of disjoint rational intervals. Then $f : \mathbb{R} \to \mathbb{R}$ is a one-dimensional piecewise affine map (PAM), if f is of the form $f(\vec{x}) = a_i\vec{x} + b_i$, where $\vec{x} \in I_i$. An example of a simple one-dimensional PAM is given in Figure 9.

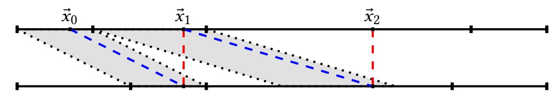


Figure 9: An example of a PAM, where the sequence $\vec{x}_0,\vec{x}_1,\vec{x}_2$ is a trajectory

In the point-to-point reachability problem for PAMs, we are given points $\vec{x}_0$ and $\vec{x}_r$ and the goal is to determine whether there is a sequence of points $\vec{x}_0,\vec{x}_1,\ldots\vec{x}_r$ such that for each $i \in \{1,\ldots,r\}$, $\vec{x}_i = f(\vec{x}_{i-1})$. In the interval-to-interval reachability problem we are given sub-intervals I_1 and I_2 of the domain of f, and the goal is to determine whether some point in I_2 is reachable from some point in I_1.

Definition 5.2 (Regular PAMs [4, 5]). A one-dimensional PAM f defined on a compact domain $D(f)$ is regular if it is injective almost everywhere. That is, f is injective except a finite number of points.

The following theorem states that PCD_{2m} systems and regular PAMs are equivalent when it comes to reachability problems.

Theorem 5.3 ([4]). *The point-to-point (interval-to-interval) reachability problem for regular PAMs can be reduced to the point-to-point (edge-to-edge) reachability problem for PCD_{2m} systems, and vice versa.*

The following theorem is a corollary of Theorems 4.2 and 5.3.

Theorem 5.4. *The interval-to-interval reachability is decidable for regular PAMs.*

6 CONCLUSIONS

In this work, we showed that mortality is decidable for piecewise constant derivative systems on surfaces (Theorem 3.1). We also showed that the edge-to-edge reachability problem is decidable for such systems (Theorem 4.2), settling in this way a problem that had been open since [4, 21]. An implication of this latter result is decidability of the interval-to-interval version of reachability for regular piecewise affine maps (Theorem 5.4).

Devising approximate reachability algorithms for classes of dynamical systems where point-to-point reachability is undecidable (or unknown to be decidable) is an interesting line of research [17].

In practical situations, it is often enough to consider an approximate version of the reachability problem, where instead of asking if a point $\vec{x}_f$ is reachable from a point $\vec{x}_0$, we ask if a point in an arbitrarily small neighbourhood of $\vec{x}_f$ is reachable from a point in an arbitrarily small neighbourhood of $\vec{x}_0$.

Determining whether point-to-point reachability for PAMs is decidable is a long-standing open problem [12]. Nevertheless, we have shown that approximate reachability for regular PAMs is decidable (Theorem 5.4), since for each $\varepsilon > 0$, the ε-neighborhood of a point in the domain of a PAM is an interval.

Our work leaves open some immediate questions. One of them is a long-standing problem of the point-to-point reachability for one-dimensional (regular) PAMs [12].

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