Homotopy Manin Triples and Higher Current Algebras

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Manin triples

Usual definition:

A **Manin triple** \((\mathfrak{a}, \mathfrak{a}_+, \mathfrak{a}_-)\) of Lie algebras over \(\mathbb{C}\) consists of

- a Lie algebra \(\mathfrak{a}\) equipped with a symmetric nondegenerate invariant bilinear form
- two isotropic Lie subalgebras \(\mathfrak{a}_+, \mathfrak{a}_-\) such that

\[
\mathfrak{a} = \mathbb{C} \mathfrak{a}_+ \oplus \mathfrak{a}_-
\]

as vector spaces.
Manin triples

Equivalently:

A Manin triple \((a, a_\pm, \iota_\pm, \langle - | - \rangle)\) of Lie algebras over \(\mathbb{C}\) consists of

1. a cospan of Lie algebras
   \[
   a_+ \xrightarrow{\iota_+} a \xleftarrow{\iota_-} a_-
   \]

2. a symmetric invariant form \(\langle - | - \rangle : a \otimes a \to \mathbb{C}\) such that
   (i) the map of vector spaces \((\iota_+, \iota_-) : a_+ \oplus a_- \to a\) is an isomorphism
   (ii) the maps \(\langle \iota_\pm(-) \mid \iota_\pm(-) \rangle : a_\pm \otimes a_\pm \to \mathbb{C}\) are zero
A homotopy Manin triple \((a, a_\pm, \iota_\pm, \langle - \mid - \rangle, n)\) of differential graded (dg) Lie algebras over \(\mathbb{C}\) consists of

1. a cospan of dg Lie algebras

\[
a_+ \xrightarrow{\iota_+} a \xleftarrow{\iota_-} a_-
\]

2. a (graded) symmetric invariant form \(\langle - \mid - \rangle : a \otimes a \to \mathbb{C}[n]\) such that

(i) the map of dg vector spaces \((\iota_+, \iota_-) : a_+ \oplus a_- \to a\) is a quasi-isomorphism

(ii) the maps \(\langle \iota_\pm(-) \mid \iota_\pm(-) \rangle : a_\pm \otimes a_\pm \to \mathbb{C}[n]\) are homotopic to zero
Plan of talk

Manin triples, rational conformal blocks, vertex algebras (review)

Beyond complex dimension one?

Higher current algebras

Rectilinear Adelic Complexes

Local and global homotopy Manin triples
Manin triples: Examples

Let $g$ be simple finite-dimensional Lie algebra over $\mathbb{C}$.

1. 
   
   \[
   a = g \otimes \mathbb{C}((x))
   \]
   
   \[
   a_+ = g \otimes \mathbb{C}[[x]], \quad a_- = g \otimes x^{-1}\mathbb{C}[x^{-1}]
   \]

2. more generally

   \[
   a = g \otimes \bigoplus_{i=1}^{N} \mathbb{C}((x - a_i))
   \]

   \[
   a_+ = g \otimes \bigoplus_{i=1}^{N} \mathbb{C}[[x - a_i]], \quad a_- = g \otimes \mathbb{C}(x)_{a_1, \ldots, a_N}^\infty
   \]

   for $a_1, \ldots, a_N \in \mathbb{C}$ distinct.
Induced modules and coinvariants

- Since \( a \cong a_+ \oplus a_- \), the PBW theorem yields
  \[
  U(a) \cong U(a_-) \otimes U(a_+)
  \]
as \((U(a_-), U(a_+))\) bimodules.

- For \( M \) a module over \( a_+ \), consider induced module

  \[
  \mathcal{M} := \text{Ind}_{a_+}^a \quad M := U(a) \otimes_{U(a_+)} M
  \]

  Then \( \mathcal{M} \) is free as a \( U(a_-) \)-module

  \[
  \mathcal{M} = U(a_-) \otimes U(a_+) \otimes_{U(a_+)} M = U(a_-) \otimes M
  \]

- Space of coinvariants:

  \[
  \mathcal{M}_{a_-} := \mathcal{M} / (a_- \cdot \mathcal{M}) \\
  \cong M
  \]

In example (2.) above, this isomorphism is highly non-trivial and leads to many rich structures.
Take $g$-modules $M_i$. (Say, finite dimensional . . . or in BGG category O.)

Make them modules over $g \otimes \mathbb{C}[[x - a_i]]$ by declaring $g \otimes (x - a_i)\mathbb{C}[[x - a_i]]$ acts as 0.

Get induced modules $M_i := U(g \otimes \mathbb{C}((x - a_i))) \otimes U(g \otimes \mathbb{C}[[x - a_i]]) M_i$

$$M = M_1 \otimes \cdots \otimes M_N$$

$$M = M_1 \otimes \cdots \otimes M_N$$

Space of coinvariants

$$M \cong \mathbb{C} M/ g \otimes \mathbb{C}(x)^{a_1, \ldots, a_N}$$

is pictorially
Aside: Rational conformal blocks

- The dual space to the space of coinvariants,

\[ M^* \cong \text{Hom}_{g \otimes \mathbb{C}(x)_{a_1, \ldots, a_N}} (M, \mathbb{C}) \]

is called the space of **rational conformal blocks**
(of “chiral g-WZW theory at level 0”).

- It can be seen as a fibre of a trivial vector bundle over configuration space

\[ \text{Conf}_N(\mathbb{C}) = \mathbb{C}^N \setminus \{a_i = a_j\} \]

- (Can equip with canonical flat connection, the **KZ connection** . . . )
Special case: **Vacuum module** induced from trivial module $\mathbb{C}|0\rangle$:

$$V := \text{Ind}_{g \otimes \mathbb{C}[[x]]}^{g \otimes \mathbb{C}(x)} \mathbb{C}|0\rangle$$

Get vector space isomorphisms ("propagation of vacua")

$$\frac{M_1 \otimes M_2 \otimes M_3 \otimes V}{g \otimes \mathbb{C}(x)_{a_1, a_2, a_3, u}} \cong M \otimes \mathbb{C} \cong M \cong \frac{M_1 \otimes M_2 \otimes M_3}{g \otimes \mathbb{C}(x)_{a_1, a_2, a_3}}$$

This leads to at least **two important constructions**:
Important construction 1: \( \mathbb{V} \) as a vertex algebra

For all \( X \in \mathbb{V} \), and all \( m_1 \otimes \cdots \otimes m_N \in \mathcal{M} = \bigotimes_{i=1}^{N} \mathcal{M}_i \) have an element

\[
[m_1 \otimes \cdots \otimes m_N \otimes X] \in \mathcal{M} \otimes \mathbb{V} / \mathfrak{g} \otimes \mathbb{C}(x)^{a_1, \ldots, a_N, u} \sim \mathcal{M} / \mathfrak{g} \otimes \mathbb{C}(x)^{a_1, \ldots, a_N}
\]

Now allow the point \( u \) to move, while holding state \( X \in \mathbb{V} \) constant (in a suitable sense, e.g. by identifying local coordinates \( x - u \) at different points \( u \) using the global coordinate \( x \)).

Get a rational function of \( u \), poles at \( \{a_1, \ldots, a_N\} \).

Take \( u \) close to one of the \( a_i \) and Laurent expand…
Important construction 1: $\mathbb{V}$ as a vertex algebra

- Find that
  
  $$\iota_{u-a_i}[m_1 \otimes \cdots \otimes m_N \otimes X] = [m_1 \otimes \cdots \otimes Y_{M_i}(X, u-a_i)m_i \otimes \cdots \otimes m_N]$$
  
  where $Y_{M_i}(-, x): \mathbb{V} \to \text{Hom}(M_i, M_i((x))).$

- In particular can take $M_i$ to be another copy of $\mathbb{V}$. Get the state-field map
  
  $$Y(-, x): \mathbb{V} \to \text{Hom}(\mathbb{V}, \mathbb{V}((x))).$$

- This makes $\mathbb{V}$ into a vertex algebra, and the $M_i$ into modules over $\mathbb{V}$.

- Usual axioms for $Y(\cdot, x)$, e.g. Borcherds identity, follow from applying residue theorem to rational functions of the form
  
  $$f(u, v)[\cdots \otimes V \otimes X \otimes Y].$$
Important construction 2: Gaudin models and geometric Langlands correspondence

1. States $X \in \mathbb{V}$ go to linear operators $X(u) \in \text{End}(M_1 \otimes \cdots \otimes M_N)$.
2. Can introduce central extension
   \[
   0 \to \mathbb{C}k \to \hat{g} \to g \otimes \mathbb{C}((u)) \to 0
   \]
3. Then... [E. Frenkel, ...]

<table>
<thead>
<tr>
<th>$\mathfrak{g}$ (simple Lie algebra)</th>
<th>Local</th>
<th>Global</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\textbf{singular vectors in } \mathbb{V}$ at critical level $k = -h^\vee$</td>
<td>Coinvariants</td>
<td>commuting Hamiltonians of quantum Gaudin model</td>
</tr>
<tr>
<td>$\uparrow$</td>
<td></td>
<td>$\uparrow$ Bethe Ansatz</td>
</tr>
<tr>
<td>$L^\mathfrak{g}$ (its Langlands dual)</td>
<td>$\mathfrak{opers}$ on the formal disc $\text{Disc}_1^\times = \text{Spec } \mathbb{C}((u))$</td>
<td>$\mathfrak{opers}$ on $\mathbb{P}^1$</td>
</tr>
</tbody>
</table>
Plan of talk

Manin triples, rational conformal blocks, vertex algebras (review)

Beyond complex dimension one?

Higher current algebras

Rectilinear Adelic Complexes

Local and global homotopy Manin triples
the constructions above were all associated to complex curves

- can we generalize to complex dimensions 2 or more?

Motivations include:

1. *(broad motivation)*
   vertex algebras/chiral CFTs/holomorphic field theory in higher dimensions
   [work of B. Williams, . . . , building on Costello-Gwilliam Factorization Algebras]

2. *(specialized motivation, this talk)*
   Gaudin models for *affine* Lie algebras
   Should describe integrals of motion of integrable quantum *field* theories.
   [Feigin, Frenkel] [Vicedo] [ . . . ]
Worldsheet
\[ \times \]
Spectral Plane

Toroidal algebras?
\[ g \otimes \mathbb{C}((x)) \otimes \mathbb{C}((u)) \]

Highest weight representations?
\[
\begin{array}{cc}
-+ & ++ \\
-+ & -+ \\
++ & \mathbb{C}[[x]] \\
-+ & x^{-1}\mathbb{C}[[x^{-1}]]
\end{array}
\]
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▶ Observe

\[ \mathbb{C}[[z]] \cong \Gamma(\text{Disc}_1, \mathcal{O}) \cong \mathbb{C}((z)) \cong \Gamma(\text{Disc}_1^\times, \mathcal{O}), \]

where \( \text{Disc}_1 := \text{Spec} \mathbb{C}[[z]] \) is the **formal 1-disc**, and

where \( \text{Disc}_1^\times := \text{Disc}_1 \setminus \{\text{pt.}\} \) is the **punctured formal 1-disc**.

▶ Natural to try same in higher dimensions, but...

\[ \mathbb{C}[[w, z]] \cong \Gamma(\text{Disc}_2, \mathcal{O}) \cong \Gamma(\text{Disc}_2^\times, \mathcal{O}), \]

where \( \text{Disc}_2 := \text{Spec} \mathbb{C}[[w, z]] \) is the formal 2-disc, and

where \( \text{Disc}_2^\times := \text{Disc}_2 \setminus \{\text{pt.}\} \) is the punctured formal 2-disc

(cf. Hartog’s theorem)

▶ But there is higher sheaf cohomology!

\[ H^\bullet(\text{Disc}_2^\times, \mathcal{O}) \cong \begin{cases} 
\mathbb{C}[[w, z]] & \bullet = 0 \\
w^{-1}z^{-1}\mathbb{C}[w^{-1}, z^{-1}] & \bullet = 1 \\
0 & \text{otherwise.}
\end{cases} \]
Consider derived sections $R\Gamma^\bullet(Disc_2^\times, \mathcal{O})$.

By definition $R\Gamma^\bullet(X, \mathcal{O})$ is a cochain complex such that

$$H^\bullet(R\Gamma(X, \mathcal{O})) \cong H^\bullet(X, \mathcal{O})$$

Moreover, since $\mathcal{O}$ is a sheaf in commutative algebras,

$$R\Gamma^\bullet(X, \mathcal{O}) = \text{holim}_{U \subset X} \Gamma(X, \mathcal{O})$$

is canonically a **dg commutative algebra**, defined up to quasi-isomorphism.

Get a **dg Lie algebra**

$$\mathfrak{g} \otimes R\Gamma^\bullet(Disc_2^\times, \mathcal{O})$$

– a **higher current algebra**

Need a good model of sheaf $U \mapsto R\Gamma^\bullet(U, \mathcal{O})$

(cf Dolbeault resolution, in complex analytic setting)

Want it to be well adapted to attaching representation-theoretic data to points, but also to certain lines and flags.
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Local and global homotopy Manin triples
Adelic complexes (sketch)

Flags $F_0 \subset F_1 \subset \cdots \subset F_k$ of subschemes of a scheme $X$ form semisimplicial set $\text{Flag}_\bullet(X)$

**Example ($\mathbb{A}^1$)**

Define suitable $A : \{\text{such flags}\} \to \{\text{comm.algebras}\}$ respecting (co)boundary maps, $A : \Delta \downarrow \text{Flag}_\bullet(X) \to \text{CAlg}$. Get a semicosimplicial algebra (carefully... adelic products!) and then take cochains. Sheafify. Get a flasque resolution of $\mathcal{O}$ & hence a model of $R\Gamma^\bullet(-, \mathcal{O})$. 

[Parshin] [Beilinson]
Rectilinear spaces

\[ \mathbb{A}^2 = \text{Spec} \, \mathbb{C}[w, z] \]

\[ \text{Rect}_2 := \mathbb{A}^1 \times_{\text{Top}} \mathbb{A}^1 \]

\[ \text{Disc}_2 = \text{Spec} \, \mathbb{C}[[w, z]] \]

\[ \text{PDisc}_2 := \text{Disc}_1 \times_{\text{Top}} \text{Disc}_1 \]

Sheaf \( \mathcal{O} := \mathcal{O}_{\mathbb{A}^1} \boxtimes \mathcal{O}_{\mathbb{A}^1} \), idea:

\[ \frac{1}{w - a} \times \frac{1}{z - b} \checkmark \quad \frac{1}{w - z} \times \]

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Rectilinear adelic complex for formal polydisc $\text{PDisc}_2$

Thom-Sullivan functor $\text{Th}^\bullet$
(rough idea)
“Paint” polynomial differential forms onto the simplicial set, valued in the correct algebras.
Defines a functor
$$\text{Th}^\bullet : [\Delta, \text{CAlg}] \to \text{dgCAlg}$$
Gives models of $\mathcal{R}\Gamma^\bullet$ with dg commutative algebra structure.

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On the unpunctured polydisc $\text{PDisc}_2$,

$$\Gamma(O, \text{PDisc}_2) = \mathbb{C}[[w]] \otimes \mathbb{C}[[z]]$$

and there is no higher cohomology.
And indeed there is a deformation retract in $\text{dgVect}$

$$\mathbb{C}[[w]] \otimes \mathbb{C}[[z]] \xrightarrow{I} \text{Th}^\bullet (A_{\text{PDisc}_2}) \xrightarrow{h}$$

Proof sketch:
$I = \text{(paint on a constant 0-form)}$; $P = \text{(take pullback to the point (0,0))}$

$$h = \begin{cases} w = 0 \end{cases}$$

(0,0)

(0,0)

(z = 0)

E
Rectilinear adelic complex for punctured formal polydisc $\operatorname{PDisc}_2^\times$

$(0, 0)$

$(w = 0)$

$(z = 0)$

Flag $\bullet(\operatorname{PDisc}_2)$

$A_{\operatorname{PDisc}_2} =$

$C[[w]] \otimes C[[z]]$

$C((w)) \otimes C((z))$

$C((w)) \otimes C((z))$

$C((w)) \otimes C((z))$

$C((w)) \otimes C((z))$

$z^{-1}C[z^{-1}]$

$w^{-1}C[w^{-1}] C[[w]]$

$C[[z]]$

$\mathbb{C}$

$\mathbb{C}$

$\otimes$

$\mathbb{C}$

$\oplus$

$\mathbb{C}$

$\otimes$

$\mathbb{C}$

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Local and global homotopy Manin triples
A “local” homotopy Manin triple

Tensor with simple Lie algebra $\mathfrak{g}$ to get dg Lie algebras

$$a = \mathfrak{g} \otimes \text{Th}^\bullet(\mathcal{A}_{\text{PDisc}^2}) \simeq \mathfrak{g} \otimes \mathcal{R}\Gamma^\bullet(\mathcal{PDisc}^2, \mathcal{O})$$

$$a_+ = \mathfrak{g} \otimes \mathbb{C}[[w]] \otimes \mathbb{C}[[z]] \simeq \mathfrak{g} \otimes \text{Th}^\bullet(\mathcal{A}_{\text{PDisc}^2}) \simeq \mathfrak{g} \otimes \mathcal{R}\Gamma^\bullet(\mathcal{PDisc}^2, \mathcal{O})$$

$$a_- = \mathfrak{g} \otimes \text{Th}^\bullet(\mathcal{A}^{--})$$

Then in $\text{dgLieAlg}$

$$a_+ \longrightarrow a \longleftarrow a_-$$

and in $\text{dgVect}$,

$$a_+ \oplus a_- \longleftrightarrow a$$
Reminder of global Manin triple in complex dimension one

\[ a = g \otimes \bigoplus_{i=1}^{N} C((x - a_i)) = g \otimes \bigoplus_{i=1}^{N} \Gamma(\text{Disc}_1^\times(a_i), \mathcal{O}) \]

\[ a_+ = g \otimes \bigoplus_{i=1}^{N} C[[x - a_i]] = g \otimes \bigoplus_{i=1}^{N} \Gamma(\text{Disc}_1(a_i), \mathcal{O}) \]

\[ a_- = g \otimes C(x)_{a_1, \ldots, a_N} = g \otimes \Gamma(\mathbb{A}_1^1 \setminus \{a_1, \ldots, a_N\}, \mathcal{O})^\infty \]
A “global” homotopy Manin triple in complex dimension two
(example of 3 marked points)

\[ a = g \otimes \bigoplus_{i=1}^{3} \text{Th}^\bullet(A_{PDisc_2^\times}(w_i, z_i)) \]
\[ a_+ = g \otimes \bigoplus_{i=1}^{3} (\mathbb{C}[[w - w_i]] \otimes \mathbb{C}[[z - z_i]]) \]
\[ a_- = g \otimes \text{Th}^\bullet(A_{Rect_2(3)}) \]

where \( A_{Rect_2(3)} \) is “mostly \( \mathbb{C}(w)_{w_1, w_2, w_3} \otimes \mathbb{C}(z)_{z_1, z_2, z_3} \), but with boundary conditions”.

\[ \simeq g \otimes \bigoplus_{i=1}^{3} R\Gamma^\bullet(PDisc_2^\times (w_i, p_i), \mathcal{O}) \]
\[ \simeq g \otimes \bigoplus_{i=1}^{3} R\Gamma^\bullet(PDisc_2(w_i, z_i), \mathcal{O}) \]
\[ \simeq g \otimes R\Gamma(\text{Rect}_2 \setminus \{(w_i, z_i)\}_{i=1}^{3}, \mathcal{O})^\infty \]
A “global” homotopy Manin triple in complex dimension two

Then $a_+ \to a \leftarrow a_-$ in $\text{dgLieAlg}$ and in $\text{dgVect}$,

\[
\begin{array}{ccc}
\text{Off-diagonal poles:} & \frac{1}{w-w_2} & \frac{1}{z-z_1} \\
(w_2, z_3) & (w_1, z_3) \\
& \downarrow & \downarrow \\
& z = z_3 & z = z_3 \\
& w = w_2 & w = w_1 \\
& \downarrow & \downarrow \\
& z = z_1 & z = z_2 \\
& w = w_3 & \\
(w_3, z_1) & (w_2, z_2) \\
\end{array}
\]

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& w = w_3 & \\
& \downarrow & \downarrow \\
& z = z_1 & z = z_2 \\
& w = w_3 & \\
& \downarrow & \downarrow \\
& z = z_1 & z = z_2 \\
& w = w_3 & \\
(w_3, z_1) & (w_2, z_2) \\
\end{array}
\]
Conclusions and outlook

Main message:
Important Manin triples of Lie algebras associated to the punctured disc/complex plane have “higher” analogs, if one goes to dg Lie algebras.

Main open question:
Of the many constructions that start with such triples, which generalize to the higher setting?

- “Quantization” a la Drinfeld Yangians??
- (Shifted) central extensions?
- Higher vertex algebras?
  (and KZ equations? hypergeometric functions? quantum groups??)
- Higher Gaudin / Affine Gaudin models + integrable QFT?
- Higher analog of Feigin-Frenkel centre??